18. Power Spectrum

For a deterministic signal $x(t)$, the spectrum is well defined: If $X(\omega)$ represents its Fourier transform, i.e., if

$$X(\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t} \, dt,$$  \hspace{1cm} (18-1)

then $|X(\omega)|^2$ represents its energy spectrum. This follows from Parseval’s theorem since the signal energy is given by

$$\int_{-\infty}^{+\infty} x^2(t) \, dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X(\omega)|^2 \, d\omega = E.$$  \hspace{1cm} (18-2)

Thus $|X(\omega)|^2 \Delta \omega$ represents the signal energy in the band $(\omega, \omega + \Delta \omega)$ (see Fig 18.1).
However for stochastic processes, a direct application of (18-1) generates a sequence of random variables for every $\omega$. Moreover, for a stochastic process, $E\{|X(t)|^2\}$ represents the ensemble average power (instantaneous energy) at the instant $t$.

To obtain the spectral distribution of power versus frequency for stochastic processes, it is best to avoid infinite intervals to begin with, and start with a finite interval $(-T, T)$ in (18-1). Formally, partial Fourier transform of a process $X(t)$ based on $(-T, T)$ is given by

$$X_T(\omega) = \int_{-T}^{T} X(t)e^{-j\omega t} \, dt$$  \hspace{1cm} (18-3)

so that

$$\frac{|X_T(\omega)|^2}{2T} = \frac{1}{2T} \left| \int_{-T}^{T} X(t)e^{-j\omega t} \, dt \right|^2$$  \hspace{1cm} (18-4)

represents the power distribution associated with that realization based on $(-T, T)$. Notice that (18-4) represents a random variable for every $\omega$, and its ensemble average gives, the average power distribution based on $(-T, T)$. Thus
$$ P_T(\omega) = E \left\{ \left| \frac{X_T(\omega)}{2T} \right|^2 \right\} = \frac{1}{2T} \int_{-T}^{T} \int_{-T}^{T} E \{X(t_1)X^*(t_2)\} e^{-j\omega(t_1-t_2)} dt_1 dt_2 $$

$$ = \frac{1}{2T} \int_{-T}^{T} \int_{-T}^{T} R_{xx}(t_1,t_2)e^{-j\omega(t_1-t_2)} dt_1 dt_2 $$

(18-5)

represents the power distribution of $X(t)$ based on $(-T, T)$. For wide sense stationary (w.s.s) processes, it is possible to further simplify (18-5). Thus if $X(t)$ is assumed to be w.s.s, then $R_{xx}(t_1,t_2) = R_{xx}(t_1-t_2)$ and (18-5) simplifies to

$$ P_T(\omega) = \frac{1}{2T} \int_{-T}^{T} \int_{-T}^{T} R_{xx}(t_1-t_2)e^{-j\omega(t_1-t_2)} dt_1 dt_2. $$

Let $\tau = t_1-t_2$ and proceeding as in (14-24), we get

$$ P_T(\omega) = \frac{1}{2T} \int_{-2T}^{2T} R_{xx}(\tau)e^{-j\omega\tau} (2T-|\tau|) d\tau $$

$$ = \int_{-2T}^{2T} R_{xx}(\tau)e^{-j\omega\tau} (1-|\tau|/2T) d\tau \geq 0 $$

(18-6)

to be the power distribution of the w.s.s. process $X(t)$ based on $(-T, T)$. Finally letting $T \rightarrow \infty$ in (18-6), we obtain
The power spectral density $S_{xx}(\omega)$ of the w.s.s process $X(t)$ is defined as

$$S_{xx}(\omega) = \lim_{T \to \infty} P_T(\omega) = \int_{-\infty}^{+\infty} R_{xx}(\tau)e^{-j\omega \tau} d\tau \geq 0 \quad (18-7)$$

to be the *power spectral density* of the w.s.s process $X(t)$. Notice that

$$R_{xx}(\omega) \xleftrightarrow{F.T} S_{xx}(\omega) \geq 0. \quad (18-8)$$
i.e., the autocorrelation function and the power spectrum of a w.s.s process form a Fourier transform pair, a relation known as the **Wiener-Khinchin Theorem**. From (18-8), the inverse formula gives

$$R_{xx}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{xx}(\omega)e^{j\omega \tau} d\omega \quad (18-9)$$

and in particular for $\tau = 0$, we get

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{xx}(\omega)d\omega = R_{xx}(0) = E\{|X(t)|^2\} = P, \quad \text{the total power.} \quad (18-10)$$

From (18-10), the area under $S_{xx}(\omega)$ represents the total power of the process $X(t)$, and hence $S_{xx}(\omega)$ truly represents the power spectrum. (Fig 18.2).
The nonnegative-definiteness property of the autocorrelation function in (14-8) translates into the “nonnegative” property for its Fourier transform (power spectrum), since from (14-8) and (18-9)

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j^* R_{xx}(t_i - t_j) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j^* \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{xx}(\omega)e^{j\omega(t_i - t_j)} d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{xx}(\omega) \left| \sum_{i=1}^{n} a_i e^{j\omega t_i} \right|^2 d\omega \geq 0. \quad (18-11)$$

From (18-11), it follows that

$$R_{xx}(\tau) \text{ nonnegative-definite } \iff S_{xx}(\omega) \geq 0. \quad (18-12)$$

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If $X(t)$ is a real w.s.s process, then $R_{XX}(\tau) = R_{XX}(-\tau)$ so that

$$S_{XX}(\omega) = \int_{-\infty}^{+\infty} R_{XX}(\tau)e^{-j\omega\tau} d\tau$$

$$= \int_{-\infty}^{+\infty} R_{XX}(\tau)\cos \omega \tau d\tau$$

$$= 2\int_{0}^{\infty} R_{XX}(\tau)\cos \omega \tau d\tau = S_{XX}(-\omega) \geq 0 \quad (18-13)$$

so that the power spectrum is an even function, (in addition to being real and nonnegative).
\[
\int_{-T}^{T} \int_{-T}^{T} R_{xx}(t_1 - t_2) dt_1 dt_2.
\]

**Comment on Slide 3:**
As \( t_1, t_2 \) varies from \(-T\) to \(+T\), \( \tau = t_1 - t_2 \) varies from \(-2T\) to \(+2T\). Moreover, \( R_{xx}(\tau) \) is a constant over the shaded region, whose area is given by

\[
\frac{1}{2} (2T - \tau)^2 - \frac{1}{2} (2T - \tau - d\tau)^2 = (2T - \tau)d\tau
\]

and hence the above integral reduces to

\[
\int_{-2T}^{2T} R_{xx}(\tau)(2T - |\tau|) d\tau = \frac{1}{2T} \int_{-2T}^{2T} R_{xx}(\tau)(1 - \frac{|\tau|}{2T}) d\tau.
\]

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Power Spectra and Linear Systems

If a w.s.s process $X(t)$ with autocorrelation function $R_{xx}(\tau) \leftrightarrow S_{xx}(\tau) \geq 0$ is applied to a linear system with impulse response $h(t)$, then the cross correlation function $R_{xy}(\tau)$ and the output autocorrelation function $R_{yy}(\tau)$ are given by (14-40)-(14-41). From there

$$R_{xy}(\tau) = R_{xx}(\tau) * h^*(-\tau), \quad R_{yy}(\tau) = R_{xx}(\tau) * h^*(-\tau) * h(\tau).$$  (18-14)

But if

$$f(t) \leftrightarrow F(\omega), \quad g(t) \leftrightarrow G(\omega)$$  (18-15)

Then

$$f(t) * g(t) \leftrightarrow F(\omega)G(\omega)$$  (18-16)

since

$$\mathcal{F}\{f(t) * g(t)\} = \int_{-\infty}^{+\infty} f(t) * g(t)e^{-j\omega t} dt$$
\[ \mathcal{F}\{f(t) * g(t)\} = \int_{-\infty}^{+\infty} \left( \int_{-\infty}^{+\infty} f(\tau) g(t-\tau)d\tau \right) e^{-j\omega t} dt \]

\[ = \int_{-\infty}^{+\infty} f(\tau) e^{-j\omega \tau} d\tau \int_{-\infty}^{+\infty} g(t-\tau) e^{-j\omega (t-\tau)} d(t-\tau) \]

\[ = F(\omega)G(\omega). \]  

(18-17)

Using (18-15)-(18-17) in (18-14) we get

\[ S_{xy}(\omega) = \mathcal{F}\{R_{xx}(\omega) * h^*(-\tau)\} = S_{xx}(\omega)H^*(\omega) \]  

(18-18)

since

\[ \int_{-\infty}^{+\infty} h^*(-\tau) e^{-j\omega \tau} d\tau = \left( \int_{-\infty}^{+\infty} h(t) e^{-j\omega t} dt \right)^* = H^*(\omega), \]

where

\[ H(\omega) = \int_{-\infty}^{+\infty} h(t) e^{-j\omega t} dt \]  

(18-19)

represents the transfer function of the system, and

\[ S_{yy}(\omega) = \mathcal{F}\{R_{yy}(\tau)\} = S_{xy}(\omega)H(\omega) \]

\[ = S_{xx}(\omega) |H(\omega)|^2. \]  

(18-20)
From (18-18), the cross spectrum need not be real or nonnegative; However the output power spectrum is real and nonnegative and is related to the input spectrum and the system transfer function as in (18-20). Eq. (18-20) can be used for system identification as well.

**W.S.S White Noise Process:** If \( W(t) \) is a w.s.s white noise process, then from (14-43)

\[
R_{ww}(\tau) = q \delta(\tau) \implies S_{ww}(\omega) = q. \tag{18-21}
\]

Thus the spectrum of a white noise process is flat, thus justifying its name. **Notice that a white noise process is unrealizable since its total power is indeterminate.**

From (18-20), if the input to an unknown system in Fig 18.3 is a white noise process, then the output spectrum is given by

\[
S_{yy}(\omega) = q |H(\omega)|^2 \tag{18-22}
\]

Notice that the output spectrum captures the system transfer function characteristics entirely, and for rational systems Eq (18-22) may be used to determine the pole/zero locations of the underlying system.
**Example 18.1:** A w.s.s white noise process $W(t)$ is passed through a low pass filter (LPF) with bandwidth $B/2$. Find the autocorrelation function of the output process.

**Solution:** Let $X(t)$ represent the output of the LPF. Then from (18-22)

$$S_{xx}(\omega) = q |H(\omega)|^2 = \begin{cases} q, & |\omega| \leq B/2 \\ 0, & |\omega| > B/2 \end{cases} \quad (18-23)$$

Inverse transform of $S_{xx}(\omega)$ gives the output autocorrelation function to be

$$R_{xx}(\tau) = \int_{-B/2}^{B/2} S_{xx}(\omega)e^{j\omega\tau}d\omega = q\int_{-B/2}^{B/2} e^{j\omega\tau}d\omega$$

$$= qB \frac{\sin(B\tau/2)}{(B\tau/2)} = qB \text{sinc}(B\tau/2) \quad (18-24)$$

![Fig. 18.4](image-url)
Eq (18-23) represents colored noise spectrum and (18-24) its autocorrelation function (see Fig 18.4).

**Example 18.2:** Let

\[ Y(t) = \frac{1}{2T} \int_{t-T}^{t+T} X(\tau)d\tau \quad (18-25) \]

represent a “smoothing” operation using a moving window on the input process \( X(t) \). Find the spectrum of the output \( Y(t) \) in term of that of \( X(t) \).

**Solution:** If we define an LTI system with impulse response \( h(t) \) as in Fig 18.5, then in term of \( h(t) \), Eq (18-25) reduces to

\[ Y(t) = \int_{-\infty}^{+\infty} h(t-\tau)X(\tau)d\tau = h(t) * X(t) \quad (18-26) \]

so that

\[ S_{yy}(\omega) = S_{xx}(\omega)|H(\omega)|^2. \quad (18-27) \]

Here

\[ H(\omega) = \int_{-T}^{+T} \frac{1}{2T} e^{-j\omega t} dt = \text{sinc}(\omega T) \quad (18-28) \]

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so that

\[ S_{yy}(\omega) = S_{xx}(\omega) \text{sinc}^2(\omega T). \]  

(18-29)

**Fig 18.6**

Notice that the effect of the smoothing operation in (18-25) is to suppress the high frequency components in the input (beyond \( \pi / T \)), and the equivalent linear system acts as a low-pass filter (continuous-time moving average) with bandwidth \( 2\pi / T \) in this case.