Markov Chains
Summary

- Markov Chains
- Discrete Time Markov Chains
  - Homogeneous and non-homogeneous Markov chains
  - Transient and steady state Markov chains
- Continuous Time Markov Chains
  - Homogeneous and non-homogeneous Markov chains
  - Transient and steady state Markov chains
Markov Processes

- The definition of a Markov Process
  - The future of process $X(t)$ does not depend on its past, only on its present
    \[
    \Pr\left\{ X(t_{k+1}) = x_{k+1} \mid X(t_k) = x_k, \ldots, X(t_0) = x_0 \right\} = \Pr\left\{ X(t_{k+1}) \leq x_{k+1} \mid X(t_k) = x_k \right\}
    \]
  - Since we are dealing with “chains”, $X(t)$ can take discrete values from a finite or a countable infinite set.

- For a discrete-time Markov chain, the notation is also simplified to
  \[
  \Pr\left\{ X_{k+1} = x_{k+1} \mid X_k = x_k, \ldots, X_0 = x_0 \right\} = \Pr\left\{ X_{k+1} = x_{k+1} \mid X_k = x_k \right\}
  \]
  - Where $x_k$ is the value of the state at the $k$th step
Transition Probability

- Define the one-step transition probabilities

\[ p_{ij}(k) = \Pr\{X_{k+1} = j \mid X_k = i\} \]

- Clearly, for all \(i, k\), and all feasible transitions from state \(i\)

\[ \sum_{j \in \Gamma(i)} p_{ij}(k) = 1 \]

- Define the \(n\)-step transition probabilities

\[ p_{ij}(k, k+n) = \Pr\{X_{k+n} = j \mid X_k = i\} \]
Chapman-Kolmogorov Equations

- Using total probability
  \[ p_{ij}(k, k+n) = \sum_{r=1}^{R} \Pr\{X_{k+n} = j \mid X_u = r, X_k = i\} \Pr\{X_u = r \mid X_k = i\} \]

- Using the memoryless property of Markov chains
  \[ \Pr\{X_{k+n} = j \mid X_u = r, X_k = i\} = \Pr\{X_{k+n} = j \mid X_u = r\} \]

- Therefore, we obtain the Chapman-Kolmogorov Equation
  \[ p_{ij}(k, k+n) = \sum_{r=1}^{R} p_{ir}(k,u) p_{rj}(u, k+n), \quad k \leq u \leq k + n \]
Matrix Form

- Define the matrix
  \[ H(k, k+n) = \left[ p_{ij}(k, k+n) \right] \]

- We can re-write the Chapman-Kolmogorov Equation
  \[ H(k, k+n) = H(k, u)H(u, k+n) \]

- Choose, \( u = k+n-1 \), then
  \[ H(k, k+n) = H(k, k+n-1)H(k+n-1, k+n) = H(k, k+n-1)P(k+n-1) \]

*Forward* Chapman-Kolmogorov

One step transition probability
Matrix Form

Choose, \( u = k+1 \), then

\[
H(k, k+n) = H(k, k+1)H(k+1, k+n) = P(k)H(k+1, k+n)
\]

*Backward* Chapman-Kolmogorov

One step transition probability
Homogeneous Markov Chains

- The one-step transition probabilities are independent of time $k$.
  
  $$P(k) = P \quad \text{or} \quad \begin{bmatrix} p_{ij} \end{bmatrix} = \begin{bmatrix} \Pr\{X_{k+1} = j \mid X_k = i\} \end{bmatrix}$$

- Even though the one step transition is independent of $k$, this does not mean that the joint probability of $X_{k+1}$ and $X_k$ is also independent of $k$
  
  Note that
  
  $$\Pr\{X_{k+1} = j, X_k = i\} = \Pr\{X_{k+1} = j \mid X_k = i\} \Pr\{X_k = i\}$$
  
  $$= p_{ij} \Pr\{X_k = i\}$$
Example

Consider a two processor computer system where, time is divided into time slots and that operates as follows

- At most one job can arrive during any time slot and this can happen with probability $\alpha$.
- Jobs are served by whichever processor is available, and if both are available then the job is given to processor 1.
- If both processors are busy, then the job is lost.
- When a processor is busy, it can complete the job with probability $\beta$ during any one time slot.
- If a job is submitted during a slot when both processors are busy but at least one processor completes a job, then the job is accepted (departures occur before arrivals).

Describe the Markov Chain that describe this model.
Example: Markov Chain

For the State Transition Diagram of the Markov Chain, each transition is simply marked with the transition probability

\[ p_{00} = (1 - \alpha) \]
\[ p_{01} = \alpha \]
\[ p_{02} = 0 \]
\[ p_{10} = \beta (1 - \alpha) \]
\[ p_{11} = (1 - \beta)(1 - \alpha) + \alpha \beta \]
\[ p_{12} = \alpha (1 - \beta) \]
\[ p_{20} = \beta^2 (1 - \alpha) \]
\[ p_{21} = \beta^2 \alpha + 2 \beta (1 - \beta)(1 - \alpha) \]
\[ p_{22} = (1 - \beta)^2 + 2 \alpha \beta (1 - \beta) \]
Example: Markov Chain

Suppose that $\alpha = 0.5$ and $\beta = 0.7$, then,

$$
\begin{bmatrix}
0.5 & 0.5 & 0 \\
0.35 & 0.5 & 0.15 \\
0.245 & 0.455 & 0.3
\end{bmatrix}
$$
State Holding Times

- Suppose that at point $k$, the Markov Chain has transitioned into state $X_k = i$. An interesting question is how long it will stay at state $i$.

- Let $V(i)$ be the random variable that represents the number of time slots that $X_k = i$.

- We are interested on the quantity $\Pr\{V(i) = n\}$

\[
\Pr\{V(i) = n\} = \Pr\{X_{k+n} \neq i, X_{k+n-1} = i, \ldots, X_{k+1} = i \mid X_k = i\}
\]

\[
= \Pr\{X_{k+n} \neq i \mid X_{k+n-1} = i, \ldots, X_k = i\} \times
\]

\[
\Pr\{X_{k+n-1} = i, \ldots, X_{k+1} = i \mid X_k = i\}
\]

\[
= \Pr\{X_{k+n} \neq i \mid X_{k+n-1} = i\} \times
\]

\[
\Pr\{X_{k+n-1} = i \mid X_{k+n-2} \ldots, X_k = i\} \times
\]

\[
\Pr\{X_{k+n-2} = i, \ldots, X_{k+1} = i \mid X_k = i\}
\]
State Holding Times

\[
\Pr\{V(i) = n\} = \Pr\left\{ X_{k+n} \neq i \mid X_{k+n-1} = i \right\} \times \\
\Pr\left\{ X_{k+n-1} = i \mid X_{k+n-2}, \ldots, X_k = i \right\} \times \\
\Pr\left\{ X_{k+n-2} = i, \ldots, X_{k+1} = i \mid X_k = i \right\} = (1 - p_{ii}) \Pr\left\{ X_{k+n-1} = i \mid X_{k+n-2} = i \right\} \times \\
\Pr\left\{ X_{k+n-2} = i \mid X_{k+n-3} = i, \ldots, X_k = i \right\} \\
\Pr\left\{ X_{k+n-3} = i, \ldots, X_{k+1} = i \mid X_k = i \right\}
\]

\[
\Pr\{V(i) = n\} = (1 - p_{ii}) \sum_{n=0}^{\infty} p_{ii}^n
\]

- This is the Geometric Distribution with parameter \( p_{ii} \).
- Clearly, \( V(i) \) has the memoryless property
An interesting quantity we are usually interested in is the probability of finding the chain at various states, i.e., we define
\[ \pi_i(k) \equiv \Pr\{X_k = i\} \]

For all possible states, we define the vector
\[ \pi(k) = [\pi_0(k), \pi_1(k), \ldots] \]

Using total probability we can write
\[ \pi_j(k) = \sum_i \Pr\{X_k = j \mid X_{k-1} = i\} \Pr\{X_{k-1} = i\} \]
\[ = \sum_i p_{ij}(k) \pi_i(k-1) \]

In vector form, one can write
\[ \pi(k) = \pi(k-1) P(k) \]
Or, if homogeneous Markov Chain
\[ \pi(k) = \pi(k-1) P \]
State Probabilities Example

- Suppose that
  \[ P = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0.35 & 0.5 & 0.15 \\ 0.245 & 0.455 & 0.3 \end{bmatrix} \]
  with \( \pi(0) = [1 \ 0 \ 0] \)

- Find \( \pi(k) \) for \( k=1,2,… \)

  \[ \pi(1) = [1 \ 0 \ 0] \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0.35 & 0.5 & 0.15 \\ 0.245 & 0.455 & 0.3 \end{bmatrix} = [0.5 \ 0.5 \ 0] \]

- Transient behavior of the system: MCTransient.m

- In general, the transient behavior is obtained by solving the difference equation

  \[ \pi(k) = \pi(k-1)P \]
Classification of States

Definitions

- State $j$ is **reachable** from state $i$ if the probability to go from $i$ to $j$ in $n > 0$ steps is greater than zero (State $j$ is reachable from state $i$ if in the state transition diagram there is a path from $i$ to $j$).

- A subset $S$ of the state space $X$ is **closed** if $p_{ij} = 0$ for every $i \in S$ and $j \notin S$.

- A state $i$ is said to be **absorbing** if it is a single element closed set.

- A closed set $S$ of states is **irreducible** if any state $j \in S$ is reachable from every state $i \in S$.

- A Markov chain is said to be **irreducible** if the state space $X$ is irreducible.
Example

- **Irreducible Markov Chain**

  ![Irreducible Markov Chain Diagram]

- **Reducible Markov Chain**

  ![Reducible Markov Chain Diagram]

  - Absorbing State
  - Closed irreducible set

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Transient and Recurrent States

- **Hitting Time** \( T_{ij} = \min \{k > 0 : X_0 = i, X_k = j\} \)

- **Recurrence Time** \( T_{ii} \) is the first time that the MC returns to state \( i \).

- Let \( \rho_i \) be the probability that the state will return back to \( i \) given it starts from \( i \). Then,
  \[
  \rho_i = \sum_{k=1}^{\infty} \Pr\{T_{ii} = k\}
  \]

- The event that the MC will return to state \( i \) given it started from \( i \) is equivalent to \( T_{ii} < \infty \), therefore we can write
  \[
  \rho_i = \sum_{k=1}^{\infty} \Pr\{T_{ii} = k\} = \Pr\{T_{ii} < \infty\}
  \]

- A state is **recurrent** if \( \rho_i = 1 \) and **transient** if \( \rho_i < 1 \)
Theorems

- If a Markov Chain has finite state space, then at least one of the states is recurrent.

- If state $i$ is recurrent and state $j$ is reachable from state $i$ then, state $j$ is also recurrent.

- If $S$ is a finite closed irreducible set of states, then every state in $S$ is recurrent.
Positive and Null Recurrent States

- Let $M_i$ be the mean recurrence time of state $i$

$$M_i \equiv E[T_{ii}] = \sum_{k=1}^{\infty} k \Pr\{T_{ii} = k\}$$

- A state is said to be **positive recurrent** if $M_i < \infty$. If $M_i = \infty$ then the state is said to be **null-recurrent**.

- Theorems
  - If state $i$ is positive recurrent and state $j$ is reachable from state $i$ then, state $j$ is also positive recurrent.
  - If $S$ is a closed irreducible set of states, then every state in $S$ is positive recurrent or, every state in $S$ is null recurrent, or, every state in $S$ is transient.
  - If $S$ is a finite closed irreducible set of states, then every state in $S$ is positive recurrent.
Example

Transient States

Recurrent States

Positive Recurrent States
Periodic and Aperiodic States

- Suppose that the structure of the Markov Chain is such that state $i$ is visited after a number of steps that is an integer multiple of an integer $d > 1$. Then the state is called periodic with period $d$.
- If no such integer exists (i.e., $d = 1$) then the state is called aperiodic.
- Example

\[
P = \begin{bmatrix}
0 & 1 & 0 \\
0.5 & 0 & 0.5 \\
0 & 1 & 0
\end{bmatrix}
\]

Periodic State $d = 2$
Steady State Analysis

- Recall that the probability of finding the MC at state $i$ after the $k$th step is given by
  \[ \pi_i(k) \equiv \Pr\{X_k = i\} \quad \pi(k) = [\pi_0(k), \pi_1(k), \ldots] \]

- An interesting question is what happens in the “long run”, i.e.,
  \[ \pi_i \equiv \lim_{k \to \infty} \pi_i(k) \]

- This is referred to as steady state or equilibrium or stationary state probability

Questions:
- Do these limits exists?
- If they exist, do they converge to a legitimate probability distribution, i.e., \[ \sum \pi_i = 1 \]
- How do we evaluate $\pi_j$, for all $j$. 

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Steady State Analysis

- Recall the recursive probability
  \[ \pi(k+1) = \pi(k)P \]
- If steady state exists, then \( \pi(k+1) \approx \pi(k) \), and therefore the steady state probabilities are given by the solution to the equations
  \[ \pi = \pi P \quad \text{and} \quad \sum \pi_i = 1 \]
- If an Irreducible Markov Chain the presence of periodic states prevents the existence of a steady state probability
- Example: *periodic.m*

\[
P = \begin{bmatrix}
0 & 1 & 0 \\
0.5 & 0 & 0.5 \\
0 & 1 & 0
\end{bmatrix} \quad \pi(0) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}
\]
Steady State Analysis

**THEOREM**: In an irreducible aperiodic Markov chain consisting of positive recurrent states a unique stationary state probability vector $\pi$ exists such that $\pi_j > 0$ and

$$\pi_j = \lim_{k \to \infty} \pi_j(k) = \frac{1}{M_j}$$

where $M_j$ is the mean recurrence time of state $j$

- The steady state vector $\pi$ is determined by solving
  $$\pi = \pi P$$
  and
  $$\sum_i \pi_i = 1$$

- Ergodic Markov chain.
Birth-Death Example

Thus, to find the steady state vector $\pi$ we need to solve

$$\pi = \pi P \quad \text{and} \quad \sum_i \pi_i = 1$$
Birth-Death Example

- In other words
  \[ \pi_0 = \pi_0 p + \pi_1 p \]
  \[ \pi_j = \pi_{j-1} (1 - p) + \pi_{j+1} p, \quad j = 1, 2, \ldots \]

- Solving these equations we get
  \[ \pi_1 = \frac{1 - p}{p} \pi_0 \]
  \[ \pi_2 = \left( \frac{1 - p}{p} \right)^2 \pi_0 \]

- In general
  \[ \pi_j = \left( \frac{1 - p}{p} \right)^j \pi_0 \]

- Summing all terms we get
  \[ \pi_0 \sum_{i=0}^{\infty} \left( \frac{1 - p}{p} \right)^i = 1 \Rightarrow \pi_0 = \frac{1}{\sum_{i=0}^{\infty} \left( \frac{1 - p}{p} \right)^i} \]
Birth-Death Example

Therefore, for all states \( j \) we get

\[
\pi_j = \left( \frac{1-p}{p} \right)^j \frac{1}{\sum_{i=0}^{\infty} \left( \frac{1-p}{p} \right)^i}
\]

1. If \( p < 1/2 \), then

\[
\sum_{i=0}^{\infty} \left( \frac{1-p}{p} \right)^i = \infty
\]

\( \Rightarrow \pi_j = 0, \) for all \( j \)

All states are transient

2. If \( p > 1/2 \), then

\[
\sum_{i=0}^{\infty} \left( \frac{1-p}{p} \right)^i = \frac{p}{2p-1} > 0
\]

\( \Rightarrow \pi_j = \frac{2p-1}{p} \left( \frac{1-p}{p} \right)^j, \) for all \( j \)

All states are positive recurrent
Birth-Death Example

- If $p=1/2$, then
  \[
  \sum_{i=0}^{\infty} \left( \frac{1-p}{p} \right)^i = \infty
  \]

  \[
  \Rightarrow \pi_j = 0, \quad \text{for all } j
  \]

  All states are *null recurrent*
Reducible Markov Chains

- In steady state, we know that the Markov chain will eventually end in an irreducible set and the previous analysis still holds, or an absorbing state.
- The only question that arises, in case there are two or more irreducible sets, is the probability it will end in each set.
Suppose we start from state $i$. Then, there are two ways to go to $S$.
- In one step or
- Go to $r \in T$ after $k$ steps, and then to $S$.

Define
\[
\rho_i(S) = \Pr\{X_k \in S \mid X_0 = i\}, \ k = 1, 2, \ldots
\]
Reducible Markov Chains

- First consider the one-step transition
  \[
  \Pr\{X_1 \in S \mid X_0 = i\} = \sum_{j \in S} p_{ij}
  \]

- Next consider the general case for \(k=2,3,\ldots\)
  \[
  \Pr\{X_k \in S, X_{k-1} = r_{k-1} \in T, \ldots, X_1 = r \in T \mid X_0 = i\} = \\
  = \Pr\{X_k \in S, X_{k-1} = r_{k-1} \in T, \ldots, X_1 = r \in T, X_0 = i\} \\
  \times \Pr\{X_1 = r \in T \mid X_0 = i\} = \\
  = \Pr\{X_k \in S, X_{k-1} = r_{k-1} \in T, \ldots, X_1 = r \in T\} p_{ir} \\
  \Rightarrow \rho_i(S) = \sum_{j \in S} p_{ij} + \sum_{r \in T} \rho_r(S) p_{ir}
  \]
Continuous-Time Markov Chains

- In this case, transitions can occur at any time.
- Recall the Markov (memoryless) property:
  \[
  \Pr\{X(t_{k+1}) = x_{k+1} \mid X(t_k) = x_k, \ldots, X(t_0) = x_0\} = \Pr\{X(t_{k+1}) = x_{k+1} \mid X(t_k) = x_k\}
  \]
  where \( t_1 < t_2 < \ldots < t_k \).
- Recall that the Markov property implies that:
  - \( X(t_{k+1}) \) depends only on \( X(t_k) \) (state memory).
  - It does not matter how long the state at \( X(t_k) \) (age memory).
- The transition probabilities now need to be defined for every time instant as \( p_{ij}(t) \), i.e., the probability that the MC transitions from state \( i \) to \( j \) at time \( t \).
Transition Function

- Define the transition function
  \[ p_{ij}(s,t) \equiv \Pr \{ X(t) = j \mid X(s) = i \}, \quad s \leq t \]

- The continuous-time analogue of the Chapman-Kolmokorov equation is
  \[ p_{ij}(s,t) \equiv \sum_r \Pr \{ X(t) = j \mid X(u) = r, X(s) = i \} \Pr \{ X(u) = r \mid X(s) = i \} \]

- Using the memoryless property
  \[ p_{ij}(s,t) \equiv \sum_r \Pr \{ X(t) = j \mid X(u) = r \} \Pr \{ X(u) = r \mid X(s) = i \} \]

- Define \( H(s,t) = [p_{ij}(s,t)], \, i,j=1,2,\ldots \) then
  \[ H(s,t) = H(s,u)H(u,t), \quad s \leq u \leq t \]

- Note that \( H(s,s) = I \)
Transition Rate Matrix

Consider the Chapman-Kolmogorov for $s \leq t \leq t + \Delta t$

$$H(s, t + \Delta t) = H(s, t)H(t, t + \Delta t)$$

Subtracting $H(s, t)$ from both sides and dividing by $\Delta t$

$$\frac{H(s, t + \Delta t) - H(s, t)}{\Delta t} = \frac{H(s, t)(H(t, t + \Delta t) - I)}{\Delta t}$$

Taking the limit as $\Delta t \to 0$

$$\frac{\partial H(s, t)}{\partial t} = H(s, t)Q(t)$$

where the transition rate matrix $Q(t)$ is given by

$$Q(t) = \lim_{\Delta t \to 0} \frac{H(t, t + \Delta t) - I}{\Delta t}$$
Homogeneous Case

In the homogeneous case, the transition functions do not depend on $s$ and $t$, but only on the difference $t-s$ thus

$$p_{ij}(s, t) = p_{ij}(t - s)$$

It follows that

$$H(s, t) = H(t - s) \equiv P(\tau)$$

and the transition rate matrix

$$Q(t) = \lim_{\Delta t \to 0} \frac{H(t, t + \Delta t) - I}{\Delta t} = \lim_{\Delta t \to 0} \frac{H(\Delta t) - I}{\Delta t} = Q, \quad \text{constant}$$

Thus

$$\frac{\partial P(t)}{\partial t} = P(t)Q \quad \text{with} \quad p_{ij}(0) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \quad \Rightarrow \quad P(t) = e^{Qt}$$
State Holding Time

All Markov processes share the interesting property that the time it takes for a change of state (sojourn time) is an exponentially-distributed random variable. To see this, let $\tau_i$ represent the waiting time for a change of state for a Markov process $x(t)$, given that it is in state $e_i$ at time $t_0$. If $\tau_i > s$, then the process will be in the same state $e_i$ at time $t_0 + s$ as at $t_0$, and (being a Markov process) its subsequent behavior is independent of $s$. Hence

$$P\{\tau_i > t + s \mid \tau_i > s\} = P\{\tau_i > t\} \overset{\Delta}{=} \varphi_i(t)$$

(16-8)
State Holding Time

represents the probability that the event \( \{ \tau_i > t + s \} \) given that \( \{ \tau_i > s \} \). But

\[
\varphi_i(t + s) = P\{ \tau_i > t + s \} = P\{ \tau_i > t + s, \tau_i > s \} = P\{ \tau_i > t + s \mid \tau_i > s \} P\{ \tau_i > s \} = \varphi_i(t) \varphi_i(s)
\]

or

\[
\log \varphi_i(t + s) = \log \varphi_i(t) + \log \varphi_i(s)
\] (16-9)

Notice that the only function that satisfies (16-9) for arbitrary \( t \) and \( s \) is either of the form \( ct \), where \( c \) is a constant or unbounded in every interval. Thus

\[
\log \varphi_i(t) = -\lambda_i t \quad \varphi_i(t) = P\{ \tau_i > t \} = e^{-\lambda_i t} \quad t \geq 0
\]

or

\[
F_{\tau_i}(t) = P\{ \tau_i \leq t \} = 1 - e^{-\lambda_i t} \quad t \geq 0
\] (16-10)
Transition Rate Matrix $Q$.

- Recall that  
  $$ \frac{\partial P(t)}{\partial t} = P(t) Q $$

- Evaluating this at $t = 0$, we have $P(0) = I$ and then  
  $$ \left. \frac{\partial P(t)}{\partial t} \right|_{t=0} = Q \Rightarrow \left. \frac{\partial p_{ij}(t)}{\partial t} \right|_{t=0} = q_{ij} $$

- If $i \neq j$, $\tau$: exponential residual lifetime  
  $$ p_{ij}(t) = \Pr\{\tau < t\} = 1 - e^{-\lambda_{ij}t} \Rightarrow \left. \frac{\partial p_{ij}(t)}{\partial t} \right|_{t=0} = q_{ij} = \lambda_{ij} e^{\lambda_{ij}t} \bigg|_{\tau=0} = \lambda_{ij} $$

- In other words $q_{ij}$ is the rate of the Poisson process that activates the event that makes the transition from $i$ to $j$. 

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Transition Rate Matrix $Q$.

- If $i = j$, $\tau$: exponential residual lifetime

$$p_{ii}(t) = \Pr\{\tau > t\} = e^{-\lambda_{ii}t} \Rightarrow \left. \frac{\partial p_{ii}(t)}{\partial t} \right|_{t=0} = q_{ii} = -\lambda_{ii} e^{\lambda_{ii}t} \bigg|_{t=0} = -\lambda_{ii}$$

$$\left. \frac{\partial p_{ii}(t)}{\partial t} \right|_{t=0} = q_{ii} \Leftrightarrow \frac{\partial}{\partial t} \left[ 1 - p_{ii}(t) \right] \bigg|_{t=0} = -q_{ii} = \lambda_{ii}$$

- Note that for each row $i$, the sum

$$\sum_j p_{ij}(t) = 1 \Rightarrow \sum_j q_{ij} = 0$$
Transition Probabilities $\mathbf{P}$.

- Suppose that state transitions occur at random points in time $T_1 < T_2 < \ldots < T_k < \ldots$
- Let $X_k$ be the state after the transition at $T_k$
- Define
  \[ P_{ij} = \Pr\{X_{k+1} = j \mid X_k = i\} \]
- Recall that in the case of the superposition of two or more Poisson processes, the probability that the next event is from process $j$ is given by $\lambda_j/\Lambda$.
- In this case, we have
  \[ P_{ij} = \frac{q_{ij}}{-q_{ii}}, \quad i \neq j \quad \text{and} \quad P_{ii} = 0 \]
Example

- Assume a computer system where jobs arrive according to a Poisson process with rate $\lambda$.
- Each job is processed using a First In First Out (FIFO) policy.
- The processing time of each job is exponential with rate $\mu$.
- The computer has buffer to store up to two jobs that wait for processing.
- Jobs that find the buffer full are lost.
- Draw the state transition diagram.
- Find the Rate Transition Matrix $Q$.
- Find the State Transition Matrix $P$.
Example

The rate transition matrix is given by

\[
Q = \begin{bmatrix}
-\lambda & \lambda & 0 & 0 \\
\mu & -(\lambda + \mu) & \lambda & 0 \\
0 & \mu & -(\lambda + \mu) & \lambda \\
0 & 0 & \mu & -\mu
\end{bmatrix}
\]

The state transition matrix is given by

\[
P = \frac{1}{(\lambda + \mu)} \begin{bmatrix}
0 & (\lambda + \mu) & 0 & 0 \\
\mu & 0 & \lambda & 0 \\
0 & \mu & 0 & \lambda \\
0 & 0 & (\lambda + \mu)^{43} & 0
\end{bmatrix}
\]
State Probabilities and Transient Analysis

- Similar to the discrete-time case, we define
  \[ \pi_j(t) \equiv \Pr\{X(t) = j\} \]

- In vector form \( \pi(t) = [\pi_1(t), \pi_2(t),...] \)

- With initial probabilities \( \pi(0) = [\pi_1(0), \pi_2(0),...] \)

- Using our previous notation (for homogeneous MC)
  \[ \pi(t) = \pi(0) P(t) = \pi(0) e^{Qt} \]

- Differentiating with respect to \( t \) gives us more “inside”
  \[ \frac{d\pi(t)}{dt} = \pi(t) Q \]
  \[ \iff \frac{d\pi_j(t)}{dt} = q_{jj}\pi_j(t) + \sum_{i \neq j} q_{ij}\pi_i(t) \]

Note: \( (e^{Qt})' = Qe^{Qt} = e^{Qt}Q \)
We view $\pi_j(t)$ as the level of a “probability fluid” that is stored at each node $j$ (0=empty, 1=full).

\[
\frac{d\pi_j(t)}{dt} = q_{jj}\pi_j(t) + \sum_{i \neq j} q_{ij}\pi_i(t) = -\sum_{r \neq j} q_{jr}\pi_j(t) + \sum_{i \neq j} q_{ij}\pi_i(t)
\]

\begin{align*}
\text{Change in the probability fluid} \\
\text{Inflow} \quad q_{ij} \quad q_{jr} \\
\text{Outflow} \quad \text{Outflow} \\
\text{Inflow} \quad \text{Inflow}
\end{align*}

\[-q_{jj} = \sum_{r \neq j} q_{jr}\]
Steady State Analysis

- Often we are interested in the “long-run” probabilistic behavior of the Markov chain, i.e.,
  \[ \pi_j = \lim_{t \to \infty} \pi_j(t) \]

- These are referred to as *steady state probabilities* or *equilibrium state probabilities* or *stationary state probabilities*.

- As with the discrete-time case, we need to address the following questions:
  - Under what conditions do the limits exist?
  - If they exist, do they form legitimate probabilities?
  - How can we evaluate these limits?
Steady State Analysis

**Theorem**: In an irreducible continuous-time Markov Chain consisting of positive recurrent states, a unique stationary state probability vector $\pi$ with

$$\pi_j = \lim_{t \to \infty} \pi_j(t)$$

These vectors are independent of the initial state probability and can be obtained by solving

$$\pi Q = 0 \quad \text{and} \quad \sum_j \pi_j = 1$$

Using the “probability fluid” view

$$0 = q_{jj} \pi_j(t) + \sum_{i \neq j} q_{ij} \pi_i(t)$$

0 Change

\[0 = q_{jj} \pi_j(t) + \sum_{i \neq j} q_{ij} \pi_i(t)\]
Example

For the previous example, with the above transition function, what are the steady state probabilities

Solve

$\pi Q = \begin{bmatrix} -\lambda & \lambda & 0 & 0 \\ \mu & -(\lambda + \mu) & \lambda & 0 \\ 0 & \mu & -(\lambda + \mu) & \lambda \\ 0 & 0 & \mu & -\mu \end{bmatrix} = 0$

$\pi_0 + \pi_1 + \pi_2 + \pi_3 = 1$
Example

The solution is obtained

\[ -\lambda \pi_0 + \mu \pi_1 = 0 \]
\[ \Rightarrow \pi_1 = \frac{\lambda}{\mu} \pi_0 \]

\[ \lambda \pi_0 - (\lambda + \mu) \pi_1 + \mu \pi_2 = 0 \]
\[ \Rightarrow \pi_2 = \left(\frac{\lambda}{\mu}\right)^2 \pi_0 \]

\[ \lambda \pi_1 - (\lambda + \mu) \pi_2 + \mu \pi_3 = 0 \]
\[ \Rightarrow \pi_3 = \left(\frac{\lambda}{\mu}\right)^3 \pi_0 \]

\[ \pi_0 + \pi_1 + \pi_2 + \pi_3 = 1 \Rightarrow \]
\[ \pi_0 = \frac{1}{1 + \left(\frac{\lambda}{\mu}\right) + \left(\frac{\lambda}{\mu}\right)^2 + \left(\frac{\lambda}{\mu}\right)^3} \]
Birth-Death Chain

- Find the steady state probabilities
- Similarly to the previous example,

\[ Q = \begin{bmatrix}
    -\lambda_0 & \lambda_0 & 0 & \cdots \\
    \mu_1 & -(\lambda_1 + \mu_1) & \lambda_1 & \cdots \\
    0 & \mu_2 & -(\lambda_2 + \mu_2) & \cdots \\
    \vdots & \vdots & \vdots & \ddots
\end{bmatrix} \]

- And we solve

\[ \pi Q = 0 \quad \text{and} \quad \sum_{i=0}^{\infty} \pi_i = 1 \]
Example

- The solution is obtained

\[ -\lambda_0 \pi_0 + \mu_1 \pi_1 = 0 \quad \Rightarrow \pi_1 = \frac{\lambda_0}{\mu_1} \pi_0 \]

\[ \lambda_0 \pi_0 - (\lambda_1 + \mu_1) \pi_1 + \mu_2 \pi_2 = 0 \quad \Rightarrow \pi_2 = \left(\frac{\lambda_0 \lambda_1}{\mu_1 \mu_2}\right) \pi_0 \]

- In general

\[ \lambda_{j-1} \pi_{j-1} - (\lambda_j + \mu_j) \pi_j + \mu_{j+1} \pi_{j+1} = 0 \quad \Rightarrow \pi_{j+1} = \left(\frac{\lambda_0 \ldots \lambda_j}{\mu_1 \ldots \mu_{j+1}}\right) \pi_0 \]

- Making the sum equal to 1

\[ \pi_0 \left(1 + \sum_{j=1}^{\infty} \left(\frac{\lambda_0 \ldots \lambda_{j-1}}{\mu_1 \ldots \mu_j}\right)\right) = 1 \]

Solution exists if

\[ S = 1 + \sum_{j=1}^{\infty} \left(\frac{\lambda_0 \ldots \lambda_{j-1}}{\mu_1 \ldots \mu_j}\right) < \infty \]
Uniformization of Markov Chains

- In general, discrete-time models are easier to work with, and computers (that are needed to solve such models) operate in discrete-time.
- Thus, we need a way to turn continuous-time to discrete-time Markov Chains.

**Uniformization Procedure**

- Recall that the total rate out of state $i$ is $-q_{ii} = \Lambda(i)$. Pick a uniform rate $\gamma$ such that $\gamma \geq \Lambda(i)$ for all states $i$.
- The difference $\gamma - \Lambda(i)$ implies a “fictitious” event that returns the MC back to state $i$ (self loop).
Uniformization of Markov Chains

Uniformization Procedure

Let $P^U_{ij}$ be the transition probability from state $i$ to state $j$ for the discrete-time uniformized Markov Chain, then

$$P^U_{ij} = \begin{cases} \frac{q_{ij}}{\gamma} & \text{if } i \neq j \\ \frac{\gamma - \sum_{j \neq i} q_{ij}}{\gamma} & \text{if } i = j \end{cases}$$