5. Functions of a Random Variable

Let $X$ be a r.v defined on the model $(\Omega, F, P)$, and suppose $g(x)$ is a function of the variable $x$. Define

$$Y = g(X). \quad (5-1)$$

Is $Y$ necessarily a r.v? If so what is its PDF $F_Y(y)$, pdf $f_Y(y)$? Clearly if $Y$ is a r.v, then for every Borel set $B$, the set of $\{\xi\}$ for which $Y(\xi) \in B$ must belong to $F$. Given that $X$ is a r.v, this is assured if $g^{-1}(B)$ is also a Borel set, i.e., if $g(x)$ is a Borel function. In that case if $X$ is a r.v, so is $Y$, and for every Borel set $B$

$$P(Y \in B) = P(X \in g^{-1}(B)). \quad (5-2)$$
Function of a RV

\[ y = g(x), \quad x \text{ is a RV.} \]

\[ F(y) = \text{Prob}\{y \leq y\} = \text{Prob}\{g(x) \leq y\} \]

Example:
\[ y = ax + b, \quad x \sim f(x) \]

\[
F_y(y) = \text{Prob}\{ax + b \leq y\} = \text{Prob}\left\{x \leq \frac{y - b}{a}\right\}, \quad a > 0
\]
\[
= F_x \left( \frac{y - b}{a} \right), \quad a > 0
\]
\[
= 1 - F_x \left( \frac{y - b}{a} \right), \quad a < 0
\]
Example:

\[ y = x^2, \quad x \sim f(x) \]

\[ F_y(y) = \text{Prob}\{x^2 \leq y\} = \text{Prob}\{-\sqrt{y} \leq x \leq \sqrt{y}\} = \]

\[ F_x(\sqrt{y}) - F_x(-\sqrt{y}), \quad y \geq 0, \quad F_y(y) = 0, \quad y < 0 \]
Example: Hard limiter

\[ y = g(x) = \begin{cases} 
1, & x > 0 \\ 
-1, & x \leq 0 
\end{cases} \]

\[ F_y(y) = \text{Prob}\{y = 1\} = \text{Prob}\{x > 0\} = 1 - F_x(0) \]

\[ F_y(y) = \text{Prob}\{y = -1\} = \text{Prob}\{x \leq 0\} = F_x(0) \]

Example: Quantization

\[ y = g(x) = ns, \quad (n-1)s < x < ns \]

\[ \text{Prob}\{y = ns\} = \text{Prob}\{(n-1)s < x < ns\} = F_x(ns) - F_x((n-1)s) \]
PDF determination

\[ y = g(x), \quad f_y(y) = \sum_{i=1}^{n} \frac{f_x(x_i)}{|g'(x_i)|}, \]

where \( x_i \) are the roots of \( y = g(x) \).

Example:

\[ y = e^x, \quad x \sim \mathcal{N}(0; \sigma^2) \]

There is only one root: \( x = \log y \)

\[ y = e^x \Rightarrow g'(x) = e^x = y \Rightarrow f_y(y) = \frac{f_x(\log y)}{y}, \quad y \geq 0 \]
Example 5.3: Let

\[
Y = g(X) = \begin{cases} 
X - c, & X > c, \\
0, & -c < X \leq c, \\
X + c, & X \leq -c.
\end{cases}
\]
In this case

\[ P(Y = 0) = P(-c < X(\xi) \leq c) = F_X(c) - F_X(-c). \quad (5-18) \]

For \( y > 0, \) we have \( x > c, \) and \( Y(\xi) = X(\xi) - c \) so that

\[ F_Y(y) = P(Y(\xi) \leq y) = P(X(\xi) - c \leq y) \]
\[ = P(X(\xi) \leq y + c) = F_X(y + c), \quad y > 0. \quad (5-19) \]

Similarly \( y < 0, \) if \( x < -c, \) and \( Y(\xi) = X(\xi) + c \) so that

\[ F_Y(y) = P(Y(\xi) \leq y) = P(X(\xi) + c \leq y) \]
\[ = P(X(\xi) \leq y - c) = F_X(y - c), \quad y < 0. \quad (5-20) \]

Thus

\[ f_Y(y) = \begin{cases} f_X(y + c), & y > 0, \\ [F_X(c) - F_X(-c)]\delta(y), \\ f_X(y - c), & y < 0. \end{cases} \quad (5-21) \]

![Fig. 5.2](image_url)
Example 5.4: Half-wave rectifier

\[ Y = g(X); \quad g(x) = \begin{cases} x, & x > 0, \\ 0, & x \leq 0. \end{cases} \quad (5-22) \]

In this case

\[ P(Y = 0) = P(X(\xi) \leq 0) = F_X(0). \quad (5-23) \]

and for \( y > 0 \), since \( Y = X \),

\[ F_y(y) = P(Y(\xi) \leq y) = P(X(\xi) \leq y) = F_X(y). \quad (5-24) \]

Thus

\[ f_y(y) = \begin{cases} f_X(y), & y > 0, \\ F_X(0)\delta(y), & y = 0, \\ 0, & y < 0, \end{cases} = f_X(y)U(y) + F_X(0)\delta(y). \quad (5-25) \]
If $x$ is an arbitrary RV with continuous distribution $F_x(x)$ and $y = F_x(x)$ then $y$ is a uniform RV in the interval $[0, 1]$. If $0 < y < 1$ then $y = F_x(x)$ has only a single solution for $x_1$. If $0 < y < 1$ then

\[ f_y(y) = \frac{f_x(x_1)}{|g'(x_1)|} = \frac{f_x(x_1)}{f_x(x_1)} = 1, \quad 0 < y < 1 \]

If $y < 0$ or $y > 1$ then $y = F_x(x)$ has no real solution then

\[ f_y(y) = 0. \]
Example
Now, If we are given two distribution functions $F_1(x)$ and $F_2(y)$, Find a monotonically increasing function $g(x)$ such that, if $y = g(x)$ and $F_x(x) = F_1(x)$ then $F_y(y) = F_2(y)$.

Solution
We maintain that $g(x)$ must be such that $F_2[g(x)] = F_1(x)$

$$F_y(y) = \text{Prob}\{y \leq y\} = \text{Prob}\{g(x) \leq g(x)\} = \text{Prob}\{x \leq x\} = F_x(x)$$

therefore, if a particular CDF $F_y(y)$ is given then RV that with such CDF is: Because $F_y(y)$ is a uniform RV then

$$x \sim \text{Unif}[0, 1] \Rightarrow y = F_y^{-1}(x)$$
Functions of a discrete-type r.v

Suppose $X$ is a discrete-type r.v with

$$P(X = x_i) = p_i, \quad x = x_1, x_2, \ldots, x_i, \ldots$$

(5-39)

and $Y = g(X)$. Clearly $Y$ is also of discrete-type, and when $x = x_i$, $y_i = g(x_i)$, and for those $y_i$

$$P(Y = y_i) = P(X = x_i) = p_i, \quad y = y_1, y_2, \ldots, y_i, \ldots$$

(5-40)

Example 5.8: Suppose $X \sim P(\lambda)$, so that

$$P(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}, \quad k = 0,1,2,\ldots$$

(5-41)

Define $Y = X^2 + 1$. Find the p.m.f of $Y$.

Solution: $X$ takes the values $0,1,2,\ldots, k, \ldots$ so that $Y$ only takes the value $1, 2, 5, \ldots, k^2 + 1, \ldots$ and
\[ P(Y = k^2 + 1) = P(X = k) \]

so that for \( j = k^2 + 1 \)

\[ P(Y = j) = P(X = \sqrt{j - 1}) = e^{-\lambda} \frac{\lambda^{\sqrt{j-1}}}{(\sqrt{j-1})!}, \quad j = 1, 2, 5, \ldots, k^2 + 1, \ldots \quad \text{(5-42)} \]
Mean, Variance, Moments and Characteristic Functions

For a r.v $X$, its p.d.f $f_x(x)$ represents complete information about it, and for any Borel set $B$ on the $x$-axis

$$P(X(\xi) \in B) = \int_B f_x(x)dx. \quad (6-1)$$

Note that $f_x(x)$ represents very detailed information, and quite often it is desirable to characterize the r.v in terms of its average behavior. In this context, we will introduce two parameters - mean and variance - that are universally used to represent the overall properties of the r.v and its p.d.f.
For continuous RV:

\[ E(x) = \eta = \int_{-\infty}^{\infty} x f_x(x) \, dx \]

For discrete type:

\[ f_x(x) = \sum_i p_i \delta(x - x_i), \quad E(x) = \sum_i p_i x_i \]

Conditional mean:

\[ E(x | \mathcal{M}) = \int_{-\infty}^{\infty} x f(x | \mathcal{M}) \, dx \]
Mean of a function of a RV:

\[ y = g(x), \quad E(y) = \int_{-\infty}^{\infty} y f_y(y) dy = \int_{-\infty}^{\infty} g(x) f_x(x) dx \]

For continuous RV, variance:

\[ \sigma^2 = \int_{-\infty}^{\infty} (x - \eta)^2 f_x(x) dx \]

For discrete type:

\[ \sigma^2 = \sum_i p_i (x_i - \eta)^2 \]
Moments:

\[ E(x^n) = \mu_n = \int_{-\infty}^{\infty} x^n f_x(x) \, dx \]

Central moments:

\[ E\{(x - \eta)^n\} = \int_{-\infty}^{\infty} (x - \eta)^n f_x(x) \, dx \]
Mean or the Expected Value of a r.v $X$ is defined as

$$
\eta_X = \bar{X} = E(X) = \int_{-\infty}^{+\infty} x f_x(x)dx.
$$

(6-2)

If $X$ is a discrete-type r.v, then using (3-25) we get

$$
\eta_X = \bar{X} = E(X) = \int x \sum_i p_i \delta(x - x_i)dx = \sum_i x_i p_i \int \delta(x - x_i)dx
$$

$$
= \sum_i x_i p_i = \sum_i x_i P(X = x_i).
$$

(6-3)

Mean represents the average (mean) value of the r.v in a very large number of trials. For example if $X \sim U(a,b)$, then using (3-31),

$$
E(X) = \int_a^b \frac{x}{b - a}dx = \frac{1}{b - a} \frac{x^2}{2}\bigg|_a^b = \frac{b^2 - a^2}{2(b - a)} = \frac{a + b}{2}
$$

(6-4)

is the midpoint of the interval $(a,b)$. 

On the other hand if $X$ is exponential with parameter $\lambda$ as in (3-32), then

$$E(X) = \int_0^\infty \frac{x}{\lambda} e^{-x/\lambda} dx = \lambda \int_0^\infty ye^{-y} dy = \lambda,$$

(6-5)

implying that the parameter $\lambda$ in (3-32) represents the mean value of the exponential r.v.

Similarly if $X$ is Poisson with parameter $\lambda$ as in (3-45), using (6-3), we get

$$E(X) = \sum_{k=0}^\infty kP(X = k) = \sum_{k=0}^\infty ke^{-\lambda} \frac{\lambda^k}{k!} = e^{-\lambda} \sum_{k=1}^\infty k \frac{\lambda^k}{k!}$$

$$= e^{-\lambda} \sum_{k=1}^\infty \frac{\lambda^k}{(k-1)!} = \lambda e^{-\lambda} \sum_{i=0}^\infty \frac{\lambda^i}{i!} = \lambda e^{-\lambda} e^\lambda = \lambda.$$

(6-6)

Thus the parameter $\lambda$ in (3-45) also represents the mean of the Poisson r.v.
In a similar manner, if $X$ is binomial as in (3-44), then its mean is given by

$$E(X) = \sum_{k=0}^{n} kP(X = k) = \sum_{k=0}^{n} k \left(\begin{array}{c} n \\ k \end{array}\right) p^k q^{n-k} = \sum_{k=1}^{n} \frac{n!}{(n-k)!k!} p^k q^{n-k}$$

$$= \sum_{k=1}^{n} \frac{n!}{(n-k)!(k-1)!} p^k q^{n-k} = np \sum_{i=0}^{n-1} \frac{(n-1)!}{(n-i-1)!i!} p^i q^{n-i-1} = np(p+q)^{n-1} = np.$$  

(6-7)

Thus $np$ represents the mean of the binomial r.v in (3-44).

For the normal r.v in (3-29),

$$E(X) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{+\infty} xe^{-(x-\mu)^2/2\sigma^2} dx = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{+\infty} (y+\mu) e^{-y^2/2\sigma^2} dy$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{+\infty} ye^{-y^2/2\sigma^2} dy + \mu \cdot \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{+\infty} e^{-y^2/2\sigma^2} dy = \mu. \quad (6-8)$$
Mean alone will not be able to truly represent the p.d.f of any r.v. To illustrate this, consider the following scenario: Consider two Gaussian r.vs $X_1 \sim N(0,1)$ and $X_2 \sim N(0,10)$. Both of them have the same mean $\mu = 0$. However, as Fig. 6.1 shows, their p.d.fs are quite different. One is more concentrated around the mean, whereas the other one ($X_2$) has a wider spread. Clearly, we need at least an additional parameter to measure this spread around the mean!

![Figure 6.1](image-url)

(a) $\sigma^2 = 1$

(b) $\sigma^2 = 10$
For a r.v $X$ with mean $\mu$, $X - \mu$ represents the deviation of the r.v from its mean. Since this deviation can be either positive or negative, consider the quantity $(X - \mu)^2$, and its average value $E[(X - \mu)^2]$ represents the average mean square deviation of $X$ around its mean. Define

$$\sigma_x^2 \triangleq E[(X - \mu)^2] > 0. \quad (6-16)$$

With $g(X) = (X - \mu)^2$ and using (6-13) we get

$$\sigma_x^2 = \int_{-\infty}^{+\infty} (x - \mu)^2 f_X(x) dx > 0. \quad (6-17)$$

$\sigma_x^2$ is known as the variance of the r.v $X$, and its square root $\sigma_x = \sqrt{E(X - \mu)^2}$ is known as the standard deviation of $X$. Note that the standard deviation represents the root mean square spread of the r.v $X$ around its mean $\mu$. 
Expanding (6-17) and using the linearity of the integrals, we get

\[ \text{Var} (X) = \sigma_x^2 = \int_{-\infty}^{+\infty} (x^2 - 2x\mu + \mu^2) f_X(x) \, dx \]

\[ = \int_{-\infty}^{+\infty} x^2 f_X(x) \, dx - 2\mu \int_{-\infty}^{+\infty} x \, f_X(x) \, dx + \mu^2 \]

\[ = E(X^2) - \mu^2 = E(X^2) - [E(X)]^2 = X^2 - \overline{X}^2. \quad (6-18) \]

Alternatively, we can use (6-18) to compute \( \sigma_x^2 \).

Thus, for example, returning back to the Poisson r.v in (3-45), using (6-6) and (6-15), we get

\[ \sigma_x^2 = \overline{X}^2 - \overline{X}^2 = (\lambda^2 + \lambda) - \lambda^2 = \lambda. \quad (6-19) \]

Thus for a Poisson r.v, mean and variance are both equal to its parameter \( \lambda \).
To determine the variance of the normal r.v \( N(\mu, \sigma^2) \), we can use (6-16). Thus from (3-29)

\[
Var(X) = E[(X - \mu)^2] = \int_{-\infty}^{\infty} (x - \mu)^2 \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2 / 2\sigma^2} dx. \tag{6-20}
\]

To simplify (6-20), we can make use of the identity

\[
\int_{-\infty}^{\infty} f_x(x) dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2 / 2\sigma^2} dx = 1
\]

for a normal p.d.f. This gives

\[
\int_{-\infty}^{\infty} e^{-(x-\mu)^2 / 2\sigma^2} dx = \sqrt{2\pi\sigma}. \tag{6-21}
\]

Differentiating both sides of (6-21) with respect to \( \sigma \), we get

\[
\int_{-\infty}^{\infty} \frac{(x - \mu)^2}{\sigma^3} e^{-(x-\mu)^2 / 2\sigma^2} dx = \sqrt{2\pi}
\]

or

\[
\int_{-\infty}^{\infty} (x - \mu)^2 \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2 / 2\sigma^2} dx = \sigma^2. \tag{6-22}
\]
which represents the \( \text{var}(X) \) in (6-20). Thus for a normal r.v as in (3-29)

\[
\text{Var}(X) = \sigma^2
\]  

(6-23)

and the second parameter in \( N(\mu, \sigma^2) \) in fact represents the variance of the Gaussian r.v. As Fig. 6.1 shows the larger the larger the spread of the p.d.f around its mean. Thus as the variance of a r.v tends to zero, it will begin to concentrate more and more around the mean ultimately behaving like a constant.

**Moments:** As remarked earlier, in general

\[
m_n = \overline{X^n} = E(X^n), \quad n \geq 1
\]  

(6-24)

are known as the moments of the r.v \( X \), and
\[ \mu_n = E[(X - \mu)^n] \quad (6-25) \]

are known as the central moments of \( X \). Clearly, the mean \( \mu = m_1 \), and the variance \( \sigma^2 = \mu_2 \). It is easy to relate \( m_n \) and \( \mu_n \). In fact

\[
\mu_n = E[(X - \mu)^n] = E\left( \sum_{k=0}^{n} \binom{n}{k} X^k (-\mu)^{n-k} \right)
\]

\[
= \sum_{k=0}^{n} \binom{n}{k} E(X^k)(-\mu)^{n-k} = \sum_{k=0}^{n} \binom{n}{k} m_k (-\mu)^{n-k}. \quad (6-26)
\]

In general, the quantities

\[ E[(X - a)^n] \quad (6-27) \]

are known as the generalized moments of \( X \) about \( a \), and

\[ E[|X|^n] \quad (6-28) \]

are known as the absolute moments of \( X \).
For example, if $X \sim N(0, \sigma^2)$, then it can be shown that

$$E(X^n) = \begin{cases} 0, & n \text{ odd,} \\ 1 \cdot 3 \cdots (n-1)\sigma^n, & n \text{ even.} \end{cases} \quad (6-29)$$

$$E(|X|^n) = \begin{cases} 1 \cdot 3 \cdots (n-1)\sigma^n, & n \text{ even,} \\ 2^k k!\sigma^{2k+1}\sqrt{2/\pi}, & n = (2k+1), \text{ odd.} \end{cases} \quad (6-30)$$

Direct use of (6-2), (6-13) or (6-14) is often a tedious procedure to compute the mean and variance, and in this context, the notion of the characteristic function can be quite helpful.

The characteristic function of a r.v $X$ is defined as
\[ \text{Prob}\{|x - \eta| \geq \varepsilon\} \leq \frac{\sigma^2}{\varepsilon^2}, \quad \text{Tchebycheff inequality} \]

\[ \text{Prob}\{|x - \eta| \geq \varepsilon\} = \int_{|x-\eta|\geq\varepsilon} f_x(x) \, dx, \quad \text{and by definition} \]

\[ \sigma^2 = \int_{-\infty}^{\infty} (x - \eta)^2 f_x(x) \, dx \quad \text{then} \quad \sigma^2 \geq \int_{|x-\eta|\geq\varepsilon} (x - \eta)^2 f_x(x) \, dx, \]

and by assumption \(|x - \eta| \geq \varepsilon\) then

\[ \sigma^2 \geq \int_{|x-\eta|\geq\varepsilon} (x - \eta)^2 f_x(x) \, dx \geq \varepsilon^2 \int_{|x-\eta|\geq\varepsilon} f_x(x) \, dx = \]

\[ \varepsilon^2 \text{Prob}\{|x - \eta| \geq \varepsilon\} \]
Characteristic function:

\[ E(e^{-j\omega x}) = \Phi(\omega) = \int_{-\infty}^{\infty} f_x(x) e^{-j\omega x} \, dx, \quad |\Phi(\omega)| \leq \Phi(0) = 1 \]

Moment Generating function:

\[ \Phi(s) = \int_{-\infty}^{\infty} f_x(x) e^{-sx} \, dx \]

Second moment generating function:

\[ \Psi(s) = \ln \Phi(s) \]
\[ \Phi^{(n)}(s) = E\{(-1)^n x^n e^{-sx}\} \Rightarrow (-1)^n \Phi^{(n)}(0) = E\{x^n\} \]

\[ \Phi(s) = \sum_{n=0}^{\infty} (-1)^n \frac{E(x^n)}{n!} s^n, \quad s \to 0 \]

This is true if moments are finite and then the series converges absolutely near \( s = 0 \).

For continuous RV, Cumulants:

\[ \gamma_n = \frac{d^n}{ds^n} \Psi(s) \bigg|_{s=0}, \quad \Psi(s) = \sum_{n=1}^{\infty} (-1)^n \frac{\gamma_n}{n!} s^n \]
For discrete RV, Characteristic function:

\[ \Phi(\omega) = E(e^{-j\omega x}) = \sum_i p_i e^{-j\omega x_i}, \quad \text{DFT of } p_i \text{ sequence} \]

If \( n \) is of lattice type RV:

\[ \Gamma(z) = E(z^n) = \sum_{n=-\infty}^{\infty} p_n z^n \]

then \( \Gamma(1/z) \) is \( z \) transform of the sequence

\[ p_n = \text{Prob}\{n = n\} \]
Example
For Binomial, and Poisson RV find $\Gamma(z)$

Solution:

$$p_k = \binom{n}{k} p^k q^{n-k} \Rightarrow \Gamma(z) = (pz + q)^n$$

$$p_k = e^{-\lambda} \frac{\lambda^k}{k!} \Rightarrow \Gamma(z) = e^{\lambda(z-1)}$$

Moments:

$$E(k^n) = \Gamma^{(n)}(z = 1)$$
Characteristic Function

\[ \Phi_X(\omega) = E(e^{j\omega X}) = \int_{-\infty}^{+\infty} e^{j\omega x} f_X(x) dx. \]

Thus \( \Phi_X(0) = 1 \), and \(|\Phi_X(\omega)| \leq 1\) for all \( \omega \).

For discrete r.v.s the characteristic function reduces to

\[ \Phi_X(\omega) = \sum_k e^{jk\omega} P(X = k). \] (6-32)

Thus for example, if \( X \sim P(\lambda) \) as in (3-45), then its characteristic function is given by

\[ \Phi_X(\omega) = \sum_{k=0}^{\infty} e^{jk\omega} e^{-\lambda} \frac{\lambda^k}{k!} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda e^{j\omega})^k}{k!} = e^{-\lambda} e^{\lambda e^{j\omega}} = e^{\lambda(e^{j\omega} - 1)}. \] (6-33)

Similarly, if \( X \) is a binomial r.v as in (3-44), its characteristic function is given by

\[ \Phi_X(\omega) = \sum_{k=0}^{n} e^{jk\omega} \binom{n}{k} p^k q^{n-k} = \sum_{k=0}^{n} \binom{n}{k} (pe^{j\omega})^k q^{n-k} = (pe^{j\omega} + q)^n. \] (6-34)

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To illustrate the usefulness of the characteristic function of a r.v in computing its moments, first it is necessary to derive the relationship between them. Towards this, from (6-31)

\[ \Phi_X(\omega) = E(e^{jX\omega}) = E\left[ \sum_{k=0}^{\infty} \frac{(j\omega X)^k}{k!} \right] = \sum_{k=0}^{\infty} \frac{j^k E(X^k)}{k!} \omega^k \]

\[ = 1 + jE(X)\omega + j^2 \frac{E(X^2)}{2!} \omega^2 + \cdots + j^k \frac{E(X^k)}{k!} \omega^k + \cdots \quad (6-35) \]

Taking the first derivative of (6-35) with respect to \( \omega \), and letting it to be equal to zero, we get

\[ \left. \frac{\partial \Phi_X(\omega)}{\partial \omega} \right|_{\omega=0} = jE(X) \quad \text{or} \quad E(X) = \frac{1}{j} \left. \frac{\partial \Phi_X(\omega)}{\partial \omega} \right|_{\omega=0}. \quad (6-36) \]

Similarly, the second derivative of (6-35) gives

\[ E(X^2) = \frac{1}{j^2} \left. \frac{\partial^2 \Phi_X(\omega)}{\partial \omega^2} \right|_{\omega=0}. \quad (6-37) \]
and repeating this procedure $k$ times, we obtain the $k$th moment of $X$ to be

$$E(X^k) = \frac{1}{j^k} \left. \frac{\partial^k \Phi_X(\omega)}{\partial \omega^k} \right|_{\omega=0}, \quad k \geq 1. \tag{6-38}$$

We can use (6-36)-(6-38) to compute the mean, variance and other higher order moments of any r.v $X$. For example, if $X \sim P(\lambda)$, then from (6-33)

$$\frac{\partial \Phi_X(\omega)}{\partial \omega} = e^{-\lambda} e^{\lambda e^{i\omega}} \lambda e^{i\omega}, \tag{6-39}$$

so that from (6-36)

$$E(X) = \lambda, \tag{6-40}$$

which agrees with (6-6). Differentiating (6-39) one more time, we get
\[ \frac{\partial^2 \Phi_X(\omega)}{\partial \omega^2} = e^{-\lambda}(e^{\lambda e^{j\omega}}(\lambda e^{j\omega})^2 + e^{\lambda e^{j\omega}}\lambda^2 e^{j\omega}), \]

so that from (6-37)

\[ E(X^2) = \lambda^2 + \lambda, \]

which again agrees with (6-15). Notice that compared to the tedious calculations in (6-6) and (6-15), the efforts involved in (6-39) and (6-41) are very minimal.

We can use the characteristic function of the binomial r.v \( B(n, p) \) in (6-34) to obtain its variance. Direct differentiation of (6-34) gives

\[ \frac{\partial \Phi_X(\omega)}{\partial \omega} = jnp e^{j\omega}(pe^{j\omega} + q)^{n-1} \]

so that from (6-36), \( E(X) = np \) as in (6-7).
One more differentiation of (6-43) yields

\[
\frac{\partial^2 \Phi_X(\omega)}{\partial \omega^2} = j^2 np \left( e^{j\omega} (pe^{j\omega} + q)^{n-1} + (n - 1) pe^{j2\omega} (pe^{j\omega} + q)^{n-2} \right) \quad (6-44)
\]

and using (6-37), we obtain the second moment of the binomial r.v to be

\[
E(X^2) = np \left( 1 + (n - 1) p \right) = n^2 p^2 + npq. \quad (6-45)
\]

Together with (6-7), (6-18) and (6-45), we obtain the variance of the binomial r.v to be

\[
\sigma_X^2 = E(X^2) - [E(X)]^2 = n^2 p^2 + npq - n^2 p^2 = npq. \quad (6-46)
\]

To obtain the characteristic function of the Gaussian r.v, we can make use of (6-31). Thus if \( X \sim N(\mu, \sigma^2) \), then
\[ \Phi_X(\omega) = \int_{-\infty}^{+\infty} e^{j\omega x} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2} \, dx \quad \text{(Let } x - \mu = y) \]

\[ = e^{j\mu\omega} \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{+\infty} e^{j\omega y} e^{-y^2/2\sigma^2} \, dy = e^{j\mu\omega} \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{+\infty} e^{-y/2\sigma^2} (y - j2\sigma^2\omega) \, dy \]

\[ \text{(Let } y - j\sigma^2\omega = u \text{ so that } y = u + j\sigma^2\omega) \]

\[ = e^{j\mu\omega} \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{+\infty} e^{-(u+j\sigma^2\omega)(u-j\sigma^2\omega)/2\sigma^2} \, du \]

\[ = e^{j\mu\omega} e^{-\sigma^2\omega^2/2} \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{+\infty} e^{-u^2/2\sigma^2} \, du = e^{(j\mu\omega - \sigma^2\omega^2/2)}. \quad (6-47) \]

Notice that the characteristic function of a Gaussian r.v itself has the “Gaussian” bell shape. Thus if \( X \sim N(0, \sigma^2) \), then

\[ f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2}, \quad (6-48) \]

and

\[ \Phi_X(\omega) = e^{-\sigma^2\omega^2/2}. \quad (6-49) \]

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