7. Two Random Variables

In many experiments, the observations are expressible not as a single quantity, but as a family of quantities. For example to record the height and weight of each person in a community or the number of people and the total income in a family, we need two numbers.

Let $X$ and $Y$ denote two random variables (r.v) based on a probability model $(\Omega, F, P)$. Then

$$ P(x_1 < X(\xi) \leq x_2) = F_X(x_2) - F_X(x_1) = \int_{x_1}^{x_2} f_X(x) dx, $$

and

$$ P(y_1 < Y(\xi) \leq y_2) = F_Y(y_2) - F_Y(y_1) = \int_{y_1}^{y_2} f_Y(y) dy. $$
What about the probability that the pair of r.v.s \((X,Y)\) belongs to an arbitrary region \(D\)? In other words, how does one estimate, for example, \(P[(x_1 < X(\xi) \leq x_2) \cap (y_1 < Y(\xi) \leq y_2)] = ?\)

Towards this, we define the joint probability distribution function of \(X\) and \(Y\) to be

\[
F_{XY}(x, y) = P[(X(\xi) \leq x) \cap (Y(\xi) \leq y)] = P(X \leq x, Y \leq y) \geq 0,
\]

where \(x\) and \(y\) are arbitrary real numbers.

**Properties**

(i) \(F_{XY}(-\infty, y) = F_{XY}(x, -\infty) = 0, \quad F_{XY}(+\infty, +\infty) = 1.\) (7-2)

since \((X(\xi) \leq -\infty, Y(\xi) \leq y) \subset (X(\xi) \leq -\infty), \) we get
Similarly we get (ii) To prove (7-3), we note that for 
\[
F_{XY}(-\infty, y) = P(X(\xi) \leq -\infty) = 0. \quad \text{Similarly } (X(\xi) \leq +\infty, Y(\xi) \leq +\infty) = \Omega,
\]
we get \( F_{XY}(\infty, \infty) = P(\Omega) = 1. \)

\[(ii) \quad P\left(x_1 < X(\xi) \leq x_2, Y(\xi) \leq y\right) = F_{XY}(x_2, y) - F_{XY}(x_1, y). \quad (7-3)\]

\[P\left(X(\xi) \leq x, y_1 < Y(\xi) \leq y_2\right) = F_{XY}(x, y_2) - F_{XY}(x, y_1). \quad (7-4)\]

To prove (7-3), we note that for \( x_2 > x_1, \)
\[(X(\xi) \leq x_2, Y(\xi) \leq y) = (X(\xi) \leq x_1, Y(\xi) \leq y) \cup (x_1 < X(\xi) \leq x_2, Y(\xi) \leq y)\]
and the mutually exclusive property of the events on the right side gives
\[P(X(\xi) \leq x_2, Y(\xi) \leq y) = P(X(\xi) \leq x_1, Y(\xi) \leq y) + P(x_1 < X(\xi) \leq x_2, Y(\xi) \leq y)\]
which proves (7-3). Similarly (7-4) follows.
(iii) \[ P(x_1 < X(\xi) \leq x_2, y_1 < Y(\xi) \leq y_2) = F_{XY}(x_2, y_2) - F_{XY}(x_2, y_1) \]
\[ - F_{XY}(x_1, y_2) + F_{XY}(x_1, y_1). \] 

(7-5)

This is the probability that \((X,Y)\) belongs to the rectangle \(R_0\) in Fig. 7.1. To prove (7-5), we can make use of the following identity involving mutually exclusive events on the right side.

\[(x_1 < X(\xi) \leq x_2, Y(\xi) \leq y_2) = (x_1 < X(\xi) \leq x_2, Y(\xi) \leq y_1) \cup (x_1 < X(\xi) \leq x_2, y_1 < Y(\xi) \leq y_2).\]
This gives

\[ P(x_1 < X(\xi) \leq x_2, Y(\xi) \leq y_2) = P(x_1 < X(\xi) \leq x_2, Y(\xi) \leq y_1) + P(x_1 < X(\xi) \leq x_2, y_1 < Y(\xi) \leq y_2) \]

and the desired result in (7-5) follows by making use of (7-3) with \( y = y_2 \) and \( y_1 \) respectively.

**Joint probability density function (Joint p.d.f)**

By definition, the joint p.d.f of \( X \) and \( Y \) is given by

\[ f_{XY}(x, y) = \frac{\partial^2 F_{XY}(x, y)}{\partial x \partial y}. \quad (7-6) \]

and hence we obtain the useful formula

\[ F_{XY}(x, y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{XY}(u, v) \, du \, dv. \quad (7-7) \]

Using (7-2), we also get

\[ \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f_{XY}(x, y) \, dx \, dy = 1. \quad (7-8) \]

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To find the probability that \((X,Y)\) belongs to an arbitrary region \(D\), we can make use of (7-5) and (7-7). From (7-5) and (7-7)

\[
P(x < X(\xi) \leq x + \Delta x, y < Y(\xi) \leq y + \Delta y) = F_{xy}(x + \Delta x, y + \Delta y) - F_{xy}(x, y + \Delta y) - F_{xy}(x + \Delta x, y) + F_{xy}(x, y)
\]

\[
= \int_{x}^{x+\Delta x} \int_{y}^{y+\Delta y} f_{xy}(u,v) dudv = f_{xy}(x, y) \Delta x \Delta y.
\]

(7-9)

Thus the probability that \((X,Y)\) belongs to a differential rectangle \(\Delta x \Delta y\) equals \(f_{xy}(x, y) \cdot \Delta x \Delta y\), and repeating this procedure over the union of no overlapping differential rectangles in \(D\), we get the useful result

![Fig. 7.2](image)
\[ P((X, Y) \in D) = \int \int_{(x, y) \in D} f_{XY}(x, y) \, dx \, dy. \]  

(7-10)

(iv) **Marginal Statistics**

In the context of several r.v.s, the statistics of each individual ones are called marginal statistics. Thus \( F_X(x) \) is the marginal probability distribution function of \( X \), and \( f_X(x) \) is the marginal p.d.f of \( X \). It is interesting to note that all marginals can be obtained from the joint p.d.f. In fact

\[ F_X(x) = F_{XY}(x, +\infty), \quad F_Y(y) = F_{XY}(+\infty, y). \]  

(7-11)

Also

\[ f_X(x) = \int_{-\infty}^{+\infty} f_{XY}(x, y) \, dy, \quad f_Y(y) = \int_{-\infty}^{+\infty} f_{XY}(x, y) \, dx. \]  

(7-12)

To prove (7-11), we can make use of the identity

\[ (X \leq x) = (X \leq x) \cap (Y \leq +\infty) \]
so that \( F_X(x) = P(X \leq x) = P(X \leq x, Y \leq \infty) = F_{XY}(x, +\infty) \).

To prove (7-12), we can make use of (7-7) and (7-11), which gives

\[
F_X(x) = F_{XY}(x, +\infty) = \int_{-\infty}^{x} \int_{-\infty}^{+\infty} f_{XY}(u, y) \, du \, dy \tag{7-13}
\]

and taking derivative with respect to \( x \) in (7-13), we get

\[
f_X(x) = \int_{-\infty}^{+\infty} f_{XY}(x, y) \, dy. \tag{7-14}
\]

At this point, it is useful to know the formula for differentiation under integrals. Let

\[
H(x) = \int_{a(x)}^{b(x)} h(x, y) \, dy. \tag{7-15}
\]

Then its derivative with respect to \( x \) is given by

\[
\frac{dH(x)}{dx} = \frac{db(x)}{dx} h(x, b) - \frac{da(x)}{dx} h(x, a) + \int_{a(x)}^{b(x)} \frac{\partial h(x, y)}{\partial x} \, dy. \tag{7-16}
\]

Obvious use of (7-16) in (7-13) gives (7-14).
If $X$ and $Y$ are discrete r.v.s, then $p_{ij} \triangleq P(X = x_i, Y = y_j)$ represents their joint p.d.f, and their respective marginal p.d.f.s are given by

$$P(X = x_i) = \sum_j P(X = x_i, Y = y_j) = \sum_j p_{ij}$$ \hspace{1cm} (7-17)

and

$$P(Y = y_j) = \sum_i P(X = x_i, Y = y_j) = \sum_i p_{ij}$$ \hspace{1cm} (7-18)

Example 7.2: $X$ and $Y$ are said to be jointly normal (Gaussian) distributed, if their joint p.d.f has the following form:

$$f_{XY}(x, y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)} \left( \frac{(x-\mu_x)^2}{\sigma_x^2} - \frac{2\rho(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y} + \frac{(y-\mu_y)^2}{\sigma_y^2} \right)}$$

$$-\infty < x < +\infty, \ -\infty < y < +\infty, \ |\rho| < 1.$$
By direct integration, using (7-14) and completing the square in (7-23), it can be shown that

\[
f_X(x) = \int_{-\infty}^{+\infty} f_{XY}(x, y) dy = \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{(x-\mu_x)^2}{2\sigma_x^2}} \sim N(\mu_x, \sigma_x^2), \quad (7-24)
\]

and similarly

\[
f_Y(y) = \int_{-\infty}^{+\infty} f_{XY}(x, y) dx = \frac{1}{\sqrt{2\pi\sigma_y^2}} e^{-\frac{(y-\mu_y)^2}{2\sigma_y^2}} \sim N(\mu_y, \sigma_y^2), \quad (7-25)
\]

Following the above notation, we will denote (7-23) as \(N(\mu_x, \mu_y, \sigma_x^2, \sigma_y^2, \rho)\). Once again, knowing the marginals in (7-24) and (7-25) alone doesn’t tell us everything about the joint p.d.f in (7-23).

As we show below, the only situation where the marginal p.d.f.s can be used to recover the joint p.d.f is when the random variables are statistically independent.
Independence of r.v.s

Definition: The random variables $X$ and $Y$ are said to be statistically independent if the events $\{X(\xi) \in A\}$ and $\{Y(\xi) \in B\}$ are independent events for any two Borel sets $A$ and $B$ in $x$ and $y$ axes respectively. Applying the above definition to the events $\{X(\xi) \leq x\}$ and $\{Y(\xi) \leq y\}$, we conclude that, if the r.v.s $X$ and $Y$ are independent, then

$$P((X(\xi) \leq x) \cap (Y(\xi) \leq y)) = P(X(\xi) \leq x)P(Y(\xi) \leq y) \quad (7-26)$$

i.e.,

$$F_{XY}(x, y) = F_X(x)F_Y(y) \quad (7-27)$$

or equivalently, if $X$ and $Y$ are independent, then we must have

$$f_{XY}(x, y) = f_X(x)f_Y(y). \quad (7-28)$$
If $X$ and $Y$ are discrete-type r.v.s then their independence implies

$$P(X = x_i, Y = y_j) = P(X = x_i)P(Y = y_j) \quad \text{for all } i, j.$$  \hspace{1cm} (7-29)

Equations (7-26)-(7-29) give us the **procedure to test for independence**. Given $f_{xy}(x, y)$, obtain the marginal p.d.fs $f_x(x)$ and $f_y(y)$ and examine whether (7-28) or (7-29) is valid. If so, the r.v.s are independent, otherwise they are dependent.

Returning back to Example 7.1, from (7-19)-(7-22), we observe by direct verification that $f_{xy}(x, y) \neq f_x(x)f_y(y)$. Hence $X$ and $Y$ are dependent r.v.s in that case. It is easy to see that such is the case in the case of Example 7.2 also, unless $\rho = 0$. In other words, two jointly Gaussian r.v.s as in (7-23) are independent if and only if the fifth parameter $\rho = 0$. 

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Example 7.3: Given

\[ f_{XY}(x, y) = \begin{cases} 
xy^2e^{-y}, & 0 < y < \infty, \quad 0 < x < 1, \\
0, & \text{otherwise}. 
\end{cases} \]  

Determine whether \( X \) and \( Y \) are independent.

Solution:

\[ f_X(x) = \int_{0}^{\infty} f_{XY}(x, y) \, dy = x\int_{0}^{\infty} y^2 e^{-y} \, dy \]
\[ = x\left[ -2 ye^{-y}\bigg|_{0}^{\infty} + 2\int_{0}^{\infty} ye^{-y} \, dy \right] = 2x, \quad 0 < x < 1. \]  

Similarly

\[ f_Y(y) = \int_{0}^{1} f_{XY}(x, y) \, dx = \frac{y^2}{2} e^{-y}, \quad 0 < y < \infty. \]

In this case

\[ f_{XY}(x, y) = f_X(x)f_Y(y), \]

and hence \( X \) and \( Y \) are independent random variables.
8. One Function of Two Random Variables

Given two random variables $X$ and $Y$ and a function $g(x,y)$, we form a new random variable $Z$ as

$$Z = g(X,Y).$$  \hspace{1cm} (8-1)

Given the joint p.d.f $f_{xy}(x,y)$, how does one obtain $f_z(z)$, the p.d.f of $Z$? Problems of this type are of interest from a practical standpoint. For example, a receiver output signal usually consists of the desired signal buried in noise, and the above formulation in that case reduces to $Z = X + Y$. 

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It is important to know the statistics of the incoming signal for proper receiver design. In this context, we shall analyze problems of the following type:

\[
X + Y \\
\max(X, Y) \quad \rightarrow \quad X - Y \\
\min(X, Y) \quad \leftrightarrow \quad Z = g(X, Y) \quad \rightarrow \quad XY \\
\sqrt{X^2 + Y^2} \quad \rightarrow \quad X / Y \\
\tan^{-1}(X / Y)
\]

Referring back to (8-1), to start with

\[
F_Z(z) = P(Z(\xi) \leq z) = P(g(X,Y) \leq z) = P[(X,Y) \in D_z] \\
= \int \int_{x,y \in D_z} f_{XY}(x,y) \, dx \, dy ,
\]

(8-3) PILLAI
where $D_z$ in the $XY$ plane represents the region such that $g(x, y) \leq z$ is satisfied. Note that $D_z$ need not be simply connected (Fig. 8.1). From (8-3), to determine $F_z(z)$ it is enough to find the region $D_z$ for every $z$, and then evaluate the integral there.

We shall illustrate this method through various examples.
Example 8.1: $Z = X + Y$. Find $f_Z(z)$.

Solution:

$$F_Z(z) = P(X + Y \leq z) = \int_{y=-\infty}^{\infty} \int_{x=-\infty}^{z-y} f_{XY}(x, y) \, dx \, dy, \quad (8-4)$$

since the region $D_z$ of the $xy$ plane where $x + y \leq z$ is the shaded area in Fig. 8.2 to the left of the line $x + y = z$. Integrating over the horizontal strip along the $x$-axis first (inner integral) followed by sliding that strip along the $y$-axis from $-\infty$ to $+\infty$ (outer integral) we cover the entire shaded area.
We can find $f_z(z)$ by differentiating $F_z(z)$ directly. In this context, it is useful to recall the differentiation rule in (7-15) - (7-16) due to Leibnitz. Suppose

$$H(z) = \int_{a(z)}^{b(z)} h(x, z) \, dx.$$  \hspace{1cm} (8-5)

Then

$$\frac{dH(z)}{dz} = \frac{db(z)}{dz} h(b(z), z) - \frac{da(z)}{dz} h(a(z), z) + \int_{a(z)}^{b(z)} \frac{\partial h(x, z)}{\partial z} \, dx.$$  \hspace{1cm} (8-6)

Using (8-6) in (8-4) we get

$$f_z(z) = \int_{-\infty}^{+\infty} \left( \frac{\partial}{\partial z} \int_{-\infty}^{z-y} f_{xy}(x, y) \, dx \right) \, dy = \int_{-\infty}^{+\infty} \left( f_{xy}(z - y, y) - 0 + \int_{-\infty}^{z-y} \frac{\partial f_{xy}(x, y)}{\partial z} \, dx \right) \, dy$$

$$= \int_{-\infty}^{+\infty} f_{xy}(z - y, y) \, dy.$$  \hspace{1cm} (8-7)

Alternatively, the integration in (8-4) can be carried out first along the $y$-axis followed by the $x$-axis as in Fig. 8.3.
In that case

\[ F_Z(z) = \int_{x=-\infty}^{+\infty} \int_{y=-\infty}^{z-x} f_{XY}(x, y) \, dx \, dy, \quad (8-8) \]

and differentiation of (8-8) gives

\[ f_Z(z) = \frac{dF_Z(z)}{dz} = \int_{x=-\infty}^{+\infty} \left( \frac{\partial}{\partial z} \int_{y=-\infty}^{z-x} f_{XY}(x, y) \, dy \right) \, dx \]
\[ = \int_{x=-\infty}^{+\infty} f_{XY}(x, z-x) \, dx. \quad (8-9) \]

If \( X \) and \( Y \) are independent, then

\[ f_{XY}(x, y) = f_X(x) f_Y(y) \quad (8-10) \]

and inserting (8-10) into (8-8) and (8-9), we get

\[ f_Z(z) = \int_{y=-\infty}^{+\infty} f_X(z-y) f_Y(y) \, dy = \int_{x=-\infty}^{+\infty} f_X(x) f_Y(z-x) \, dx. \quad (8-11) \]
The above integral is the standard convolution of the functions $f_x(z)$ and $f_y(z)$ expressed two different ways. We thus reach the following conclusion: If two r.vs are independent, then the density of their sum equals the convolution of their density functions.
Example 8.2: Suppose $X$ and $Y$ are independent exponential r.v.s with common parameter $\lambda$, and let $Z = X + Y$.
Determine $f_Z(z)$.
Solution: We have 
\[ f_X(x) = \lambda e^{-\lambda x} U(x), \quad f_Y(y) = \lambda e^{-\lambda y} U(y), \] 
and we can make use of (13) to obtain the p.d.f of $Z = X + Y$.
\[ f_Z(z) = \int_0^z \lambda^2 e^{-\lambda x} e^{-\lambda (z-x)} \, dx = \lambda^2 e^{-\lambda z} \int_0^z \, dx = z \lambda^2 e^{-\lambda z} U(z). \] 
As the next example shows, care should be taken in using the convolution formula for r.v.s with finite range.

Example 8.3: $X$ and $Y$ are independent uniform r.v.s in the common interval $(0,1)$. Determine $f_Z(z)$, where $Z = X + Y$.
Solution: Clearly, $Z = X + Y \Rightarrow 0 < z < 2$ here, and as Fig. 8.5 shows there are two cases of $z$ for which the shaded areas are quite different in shape and they should be considered separately.
For $0 \leq z < 1$,

$$F_Z(z) = \int_0^z \int_{y=0}^{z-y} 1 \, dx \, dy = \int_0^z (z - y) \, dy = \frac{z^2}{2}, \quad 0 \leq z < 1. \quad (8-16)$$

For $1 \leq z < 2$, notice that it is easy to deal with the unshaded region. In that case

$$F_Z(z) = 1 - P(Z > z) = 1 - \int_{y=z-1}^1 \int_{x=z-y}^1 1 \, dx \, dy$$

$$= 1 - \int_{y=z-1}^1 (1 - z + y) \, dy = 1 - \frac{(2-z)^2}{2}, \quad 1 \leq z < 2. \quad (8-17)$$
Using (8-16) - (8-17), we obtain

\[ f_Z(z) = \frac{dF_Z(z)}{dz} = \begin{cases} 
z & 0 \leq z < 1, \\
2 - z & 1 \leq z < 2.
\end{cases} \quad (8-18) \]

By direct convolution of \( f_X(x) \) and \( f_Y(y) \), we obtain the same result as above. In fact, for \( 0 \leq z < 1 \) (Fig. 8.6(a))

\[ f_Z(z) = \int f_X(z-x)f_Y(x)dx = \int_{0}^{z} 1 \, dx = z. \quad (8-19) \]

and for \( 1 \leq z < 2 \) (Fig. 8.6(b))

\[ f_Z(z) = \int_{z-1}^{1} 1 \, dx = 2 - z. \quad (8-20) \]

Fig 8.6 (c) shows \( f_Z(z) \) which agrees with the convolution of two rectangular waveforms as well.
Fig. 8.6 (c)

(a) $0 \leq z < 1$

(b) $1 \leq z < 2$
Example 8.6: \( Z = X^2 + Y^2 \). Obtain \( f_z(z) \).

Solution: We have

\[
F_z(z) = P(X^2 + Y^2 \leq z) = \int \int_{x^2+y^2 \leq z} f_{xy}(x, y) \, dx \, dy.
\]

(8-32)

\[
F_z(z) = \int_{y=-\sqrt{z}}^{\sqrt{z}} \int_{x=-\sqrt{z-y^2}}^{\sqrt{z-y^2}} f_{xy}(x, y) \, dx \, dy.
\]

\[
f_z(z) = \int_{y=-\sqrt{z}}^{\sqrt{z}} \frac{1}{2\sqrt{z-y^2}} \left( f_{xy}(\sqrt{z-y^2}, y) + f_{xy}(-\sqrt{z-y^2}, y) \right) \, dy.
\]
Example 8.7: X and Y are independent normal r.v.s with zero Mean and common variance $\sigma^2$. Determine $f_Z(z)$ for $Z = X^2 + Y^2$.

Solution: Direct substitution of (8-29) with $r=0, \quad \sigma_1 = \sigma_2 = \sigma$

Into (8-34) gives

$$f_Z(z) = \int_{y=-\sqrt{z}}^{\sqrt{z}} \frac{1}{2\sqrt{z-y^2}} \left(2 \cdot \frac{1}{2\pi\sigma^2} e^{-(z-y^2+y^2)/2\sigma^2}\right) dy = \frac{e^{-z/\sigma^2}}{\pi\sigma^2} \int_0^{\sqrt{z}} \frac{1}{\sqrt{z-y^2}} \ dy$$

$$= \frac{e^{-z/\sigma^2}}{\pi\sigma^2} \int_0^{\pi/2} \frac{\sqrt{z} \cos \theta}{\sqrt{z} \cos \theta} d\theta = \frac{1}{2\sigma^2} e^{-z/2\sigma^2} U(z), \quad (8-35)$$

where we have used the substitution $y = \sqrt{z} \sin \theta$. From (8-35) we have the following result: If X and Y are independent zero mean Gaussian r.v.s with common variance $\sigma^2$, then $X^2 + Y^2$ is an exponential r.v. with parameter $2\sigma^2$.

Example 8.8: Let $Z = \sqrt{X^2 + Y^2}$. Find $f_Z(z)$.

Solution: From Fig. 8.11, the present case corresponds to a circle with radius $z^2$. Thus
\[ F_z(z) = \int_{y=-z}^{z} \int_{x=-\sqrt{z^2-y^2}}^{\sqrt{z^2-y^2}} f_{xy}(x, y) \, dx \, dy. \]

And by repeated differentiation, we obtain

\[ f_z(z) = \int_{-z}^{z} \frac{z}{\sqrt{z^2-y^2}} \left( f_{xy}(\sqrt{z^2-y^2}, y) + f_{xy}(-\sqrt{z^2-y^2}, y) \right) dy. \quad (8-36) \]

Now suppose \( X \) and \( Y \) are independent Gaussian as in Example 8.7. In that case, (8-36) simplifies to

\[
\begin{align*}
f_z(z) &= 2 \int_{0}^{z} \frac{1}{\sqrt{z^2-y^2}} \frac{e^{(z^2-y^2+y^2)/2\sigma^2}}{2\pi\sigma^2} \, dy = \frac{2z}{\pi\sigma^2} e^{-z^2/2\sigma^2} \int_{0}^{z} \frac{1}{\sqrt{z^2-y^2}} \, dy \\
&= \frac{2z}{\pi\sigma^2} e^{-z^2/2\sigma^2} \int_{0}^{\pi/2} \frac{z \cos \theta}{z \cos \theta} \, d\theta = \frac{z}{\sigma^2} e^{-z^2/2\sigma^2} U(z),
\end{align*}
\tag{8-37}
\]

which represents a Rayleigh distribution. Thus, if \( W = X + iY \), where \( X \) and \( Y \) are real, independent normal r.v.s with zero mean and equal variance, then the r.v \( |W| = \sqrt{X^2 + Y^2} \) has a Rayleigh density. \( W \) is said to be a complex Gaussian r.v with zero mean, whose real and imaginary parts are independent r.v.s. From (8-37), we have seen that its magnitude has Rayleigh distribution.
Example 8.13 (Discrete Case): Let $X$ and $Y$ be independent Poisson random variables with parameters $\lambda_1$ and $\lambda_2$ respectively. Let $Z = X + Y$. Determine the p.m.f of $Z$. 


Solution: Since $X$ and $Y$ both take integer values $\{0, 1, 2, \cdots\}$, the same is true for $Z$. For any $n = 0, 1, 2, \cdots$, $X + Y = n$ gives only a finite number of options for $X$ and $Y$. In fact, if $X = 0$, then $Y$ must be $n$; if $X = 1$, then $Y$ must be $n-1$, etc. Thus the event $\{X + Y = n\}$ is the union of $(n + 1)$ mutually exclusive events $A_k$ given by

$$A_k = \{X = k, \ Y = n-k\}, \quad k = 0, 1, 2, \cdots, n. \quad (8-55)$$

As a result

$$P(Z = n) = P(X + Y = n) = P\left(\bigcup_{k=0}^{n} (X = k, \ Y = n-k)\right)$$

$$= \sum_{k=0}^{n} P(X = k, \ Y = n-k). \quad (8-56)$$

If $X$ and $Y$ are also independent, then

$$P\left(X = k, \ Y = n-k\right) = P(X = k)P(Y = n-k)$$

and hence
Thus \( Z \) represents a Poisson random variable with parameter \( \lambda_1 + \lambda_2 \), indicating that sum of independent Poisson random variables is also a Poisson random variable whose parameter is the sum of the parameters of the original random variables.