Filter design

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Outline

- Filter types
- Analog filters
- IIR and FIR filters
  - IIR
    - Impulse invariant
    - Bilinear
    - Applications: speech equalization, ....
  - FIR
    - Windowing
    - Applications
Filter

- Any discrete-time system that modifies certain frequencies
- Frequency-selective filters pass only certain frequencies

Filters
- Lowpass
- Highpass
- Bandpass
- Bandstop
Lowpass & highpass filter

- Low frequency components are passed through the filter while the high-frequency components are attenuated.
- Keeps high-frequency components and rejects low-frequency components.
Lowpass & highpass filter

- **Passband**
  - the frequency range with the amplitude gain of the filter response being approximately unity.

- **Stopband**
  - The frequency range over which the filter magnitude response is attenuated to eliminate the input signal whose frequency components are within that range.

- **Transition band**
  - the frequency range between the passband and stopband
Lowpass & highpass filter

- $\Omega_p$: passband cutoff frequency
- $\Omega_s$: stopband cutoff frequency
- $\delta_p$: ripple (fluctuation) of the frequency response in the passband.
- $\delta_s$: ripple of the frequency response in the stopband.
Bandpass filter

- The bandpass filter attenuates both low- and high-frequency components while keeping the middle frequency components.
  - $\Omega_{sL}$: lower stopband cutoff frequency
  - $\Omega_{pL}$: lower passband cutoff frequency
  - $\Omega_{pH}$: higher passband cutoff frequency
  - $\Omega_{sH}$: higher stopband cutoff frequency
Bandstop (notch) filter

- rejects the middle-frequency components and accepts both the low- and the high-frequency components.
Continuous-time filters

- Butterworth
- Chebyshev
  - Type I
  - Type II
- Elliptic
Butterworth Lowpass Filters

- Magnitude response is maximally flat in the pass band.
- Magnitude response is monotonic in the passband and stopband.
- The magnitude-squared function is of the form

\[
|H_c(j\Omega)|^2 = \frac{1}{1 + (j\Omega / j\Omega_c)^{2N}}
\]

\[
|H_c(s)|^2 = \frac{1}{1 + (s / j\Omega_c)^{2N}}
\]
Butterworth Lowpass Filters

\[ H_c(s)H_c(-s) = \frac{1}{1+(s/j\Omega_c)^{2N}}. \]

\[ 1 + (s/j\Omega_c)^{2N} = 0; \]

\[ s_k = (-1)^{1/2N}(j\Omega_c) = \Omega_c e^{(j\pi/2N)(2k+N-1)}, \quad k = 0, 1, \ldots, 2N-1. \]

- There are 2N poles equally spaced in angle on circle of radius \( \Omega_c \).
- Poles occurs in pairs, \( s = s_k \)

![Diagram](attachment:image.png)
Chebyshev Filters

- Type I
  - Equiripple in the passband and monotonic in the stopband
  - $|H_c(j\Omega)|$ ripples between 1 and $\frac{1}{1+\epsilon^2}$ for $0 \leq \frac{\Omega}{\Omega_c} \leq 1$ and decreases monotonically for $\frac{\Omega}{\Omega_c} \geq 1$

$$|H_c(j\Omega)|^2 = \frac{1}{1 + \epsilon^2 V_N^2(\Omega / \Omega_c)} \quad V_N(x) = \cos(N \cos^{-1} x)$$
Chebyshev Filters

- Type II
  - Monotonic in the passband and equiripple in the stopband
  - One approach to design type II filter is to first design type I filter and then apply the above transformation.

\[
|H_c(j\Omega)|^2 = \frac{1}{1 + \left[ \epsilon^2 \mathcal{P}_N^2(\Omega_c/\Omega) \right]^{-1}}.
\]
Elliptic filter

- Equiripple in both passband and stopband

\[ |H_c(j\Omega)|^2 = \frac{1}{1 + \varepsilon^2 U_N^2(\Omega)} \]

- \( U_N(\Omega) \): Jacobian elliptic function
Digital filters
Ideal filters are non causal and nonrealizable
Discrete-time processing of continuous-time signals

- We already studied the use of discrete-time systems to implement a continuous-time system.
  - If the input is band limited and the sampling frequency is high to avoid aliasing, then the overall system behaves as a LTI continuous-time system.

  \[ H_{\text{eff}}(j\Omega) = \begin{cases} 
  H(e^{j\Omega T}), & |\Omega| < \pi / T, \\
  0, & |\Omega| > \pi / T.
\end{cases} \]

- If our specifications are given in continuous time we can obtain discrete-time specifications.

  \[ \omega = \Omega T \]
  \[ H(e^{j\omega}) = H_{\text{eff}} \left( j \frac{\omega}{T} \right) \quad |\omega| < \pi \]
Filter Specifications

Specifications
- **Passband**
  \[0.99 \leq |H_{\text{eff}}(j\Omega)| = 1.01 \quad 0 \leq \Omega \leq 2\pi(2000)\]
- **Stopband**
  \[|H_{\text{eff}}(j\Omega)| \leq 0.001 \quad 2\pi(3000) \leq \Omega\]

Parameters
- \(\delta_1 = 0.01\)
- \(\delta_2 = 0.001\)
- \(\Omega_p = 2\pi(2000) \rightarrow \omega_p = 2\pi(2000)(10^{-4}) = 0.4\pi\)
- \(\Omega_s = 2\pi(3000) \rightarrow \omega_s = 2\pi(3000)(10^{-4}) = 0.6\pi\)

Specs in dB
- Ideal passband gain = \(20\log(1) = 0\) dB
- Max passband gain = \(20\log(1.01) = 0.086\) dB
- Max stopband gain = \(20\log(0.001) = -60\) dB
all kinds of digital filters are implemented using FIR or IIR systems.

**Infinite impulse response (IIR)**

\[ y(n) = b_0 x(n) + b_1 x(n-1) + \cdots + b_M x(n-M) \]
\[ -a_1 y(n-1) - a_2 y(n-2) - \cdots - a_N y(n-N) \]
\[ H(z) = \frac{B(z)}{A(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}} \]

**finite impulse response (FIR)**

\[ y(n) = \sum_{i=0}^{K} b_i x(n-i) \]
\[ = b_0 x(n) + b_1 x(n-1) + b_2 x(n-2) + \cdots + b_K x(n-K) \]
\[ H(z) = \frac{Y(z)}{X(z)} = b_0 + b_1 z^{-1} + \cdots + b_K z^{-K} \]
IIR or FIR?

- Compared with FIR filter, the IIR filter
  - offers a much smaller filter size.
  - Filter operation requires a fewer number of computations.
  - Not linear phase

- FIR
  - Linear phase
  - much higher filter order than IIR filters
  - the design methods often are iterative in nature requiring computer-aided techniques.
Filter design

Filter Design Steps

- Specification
  - Problem or application specific
- Approximation of specification with a discrete-time system
  - Our focus is to go from spec to discrete-time system
- Implementation
  - Realization of discrete-time systems depends on target technology
IIR filter design
The techniques for the design of standard frequency selective discrete-time IIR filters are based on well-developed continuous-time filter design methods.

- Impulse invariant
- Bilinear transformation

Figure 11.1 Procedure for designing IIR filters from continuous-time filters.
IIR filter design using impulse invariant

- Sampling the impulse response of continuous-time filter.
  - it preserves the shape of the impulse response.
    \[
    h[n] = T_d h_c(nT_d) \Rightarrow H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} H_c\left(j\frac{\omega}{T_d} + j\frac{2\pi}{T_d} k\right)
    \]
  - If the continuous-time filter is bandlimited to
    \[
    H_c(j\Omega) = 0 \quad |\Omega| \geq \pi / T_d \Rightarrow H(e^{j\omega}) = H_c\left(j\frac{\omega}{T_d}\right) \quad |\omega| \leq \pi
    \]
- Continuous-time and discrete-time frequencies are related by the linear scaling \(\omega = \Omega T_d\).
- Because of the aliasing effect, the impulse-invariance method is only meaningful for bandlimited filters, like lowpass and bandpass filters.
Impulse Invariance of System Functions

Consider the CT system function with partial fraction expansion

$$H_c(s) = \sum_{k=1}^{N} \frac{A_k}{s - s_k}$$

Corresponding impulse response

$$h_c(t) = \begin{cases} \sum_{k=1}^{N} A_k e^{s_k t} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Impulse response of discrete-time filter

$$h[n] = T_d h_c(n T_d) = \sum_{k=1}^{N} T_d A_k e^{s_k n T_d} u[n] = \sum_{k=1}^{N} T_d A_k e^{s_k T_d} u[n]$$

Then, DT system function is

$$H(z) = \sum_{k=1}^{N} \frac{T_d A_k}{1 - e^{s_k T_d} z^{-1}}$$

Pole $s = s_k$ in s-domain transform into pole at $e^{s_k T_d}$
Not one-to-one mapping

\[
\begin{align*}
\ & s = \sigma + j\Omega \rightarrow Z = e^{\sigma T_d} e^{j\Omega T_d} \\
\ & s = \sigma + j\left(\Omega + \frac{2\pi}{T_d}\right) \rightarrow Z = e^{\sigma T_d} e^{j\left(\Omega + \frac{2\pi}{T_d}\right) T_d} = e^{\sigma T_d} e^{j\Omega T_d} \\
\ & \text{A strip of height } 2\pi/T_d \text{ is mapped into the entire } z\text{-plane.}
\end{align*}
\]
Example

- Impulse invariance applied to Butterworth
  \[ 0.89125 \leq |H(e^{j\omega})| \leq 1 \quad 0 \leq |\omega| \leq 0.2\pi \]
  \[ |H(e^{j\omega})| \leq 0.17783 \quad 0.3\pi \leq |\omega| \leq \pi \]

- Since sampling rate \( T_d \) cancels out we can assume \( T_d = 1 \)

- Map spec to continuous time
  \[ 0.89125 \leq |H(j\Omega)| \leq 1 \quad 0 \leq |\Omega| \leq 0.2\pi \]
  \[ |H(j\Omega)| \leq 0.17783 \quad 0.3\pi \leq |\Omega| \leq \pi \]

- Butterworth filter is monotonic so spec will be satisfied if
  \[ |H_c(j0.2\pi)| \geq 0.89125 \quad \text{and} \quad |H_c(j0.3\pi)| \leq 0.17783 \]
  \[ |H_c(j\Omega)|^2 = \frac{1}{1 + (j\Omega / j\Omega_c)^{2N}} \]

- Determine \( N \) and \( \Omega_c \) to satisfy these conditions
Example Cont’d

- Satisfy both constrains

\[ 1 + \left( \frac{0.2\pi}{\Omega_c} \right)^{2^N} = \left( \frac{1}{0.89125} \right)^2 \quad \text{and} \quad 1 + \left( \frac{0.3\pi}{\Omega_c} \right)^{2^N} = \left( \frac{1}{0.17783} \right)^2 \]

- Solve these equations to get

\[ N = 5.8858 \approx 6 \quad \text{and} \quad \Omega_c = 0.70474 \]

- Poles of transfer function

\[ s_k = (-1)^{1/12} (j\Omega_c) = \Omega_c e^{(j\pi/12)(2k+11)} \quad \text{for} \ k = 0,1,\ldots,11 \]

- Pole pair 1: \(-0.182 \pm j(0.679),\)
- Pole pair 2: \(-0.497 \pm j(0.497),\)
- Pole pair 3: \(-0.679 \pm j(0.182).\)
Example Cont’d

- For causality and stability, we select poles in the left half of s-plane.

![s-plane diagram]

- Pole pair 1: \(-0.182 \pm j(0.679)\),
- Pole pair 2: \(-0.497 \pm j(0.497)\),
- Pole pair 3: \(-0.679 \pm j(0.182)\).
The transfer function

$$H(s) = \frac{0.12093}{(s^2 + 0.364s + 0.4945)(s^2 + 0.9945s + 0.4945)(s^2 + 1.3585s + 0.4945)}$$

Mapping to z-domain

$$H(z) = \frac{0.2871 - 0.4466z^{-1}}{1 - 1.2971z^{-1} + 0.6949z^{-2}} + \frac{-2.1428 + 1.1455z^{-1}}{1 - 1.0691z^{-1} + 0.3699z^{-2}} + \frac{1.8557 - 0.6303z^{-1}}{1 - 0.9972z^{-1} + 0.257z^{-2}}$$
Freq. response
Bilinear transformation

- To avoid the limitations of impulse-invariance transformation caused by the aliasing effect,
- we need a one-to-one mapping from the s-plane to the z-plane.
- The *bilinear transformation* is an invertible nonlinear mapping between the s-plane and the z-plane defined by
Filter Design by Bilinear Transformation

- Avoid the aliasing problem of impulse invariance
- Map the entire $j\Omega$-axis in the $s$-plane to one revolution of the unit-circle in the $z$-plane.
  - Nonlinear transformation
  - Frequency response subject to warping
- Bilinear transformation
  \[
  s = \frac{2}{T_d} \left( \frac{1 - z^{-1}}{1 + z^{-1}} \right)
  \]
- Transformed system function
  \[
  H(z) = H_c \left[ \frac{2}{T_d} \left( \frac{1 - z^{-1}}{1 + z^{-1}} \right) \right]
  \]
- We can solve the transformation for $z$ as
  \[
  z = \frac{1 + \left( T_d / 2 \right) s}{1 - \left( T_d / 2 \right) s} = \frac{1 + \sigma T_d / 2 + j\Omega T_d / 2}{1 - \sigma T_d / 2 - j\Omega T_d / 2}
  \]
  \[
  s = \sigma + j\Omega
  \]
Maps the left-half s-plane into the inside of the unit-circle in z

Causal Stable continuous-time filter map into the causal stable discrete-time filter
Bilinear Transformation

- On the unit circle the transform becomes

\[ z = \frac{1 + j\Omega T_d / 2}{1 - j\Omega T_d / 2} = e^{j\omega} \]

- To derive the relation between \( \omega \) and \( \Omega \)

\[
s = \frac{2}{T_d} \left( \frac{1 - e^{-j\omega}}{1 + e^{-j\omega}} \right) = \sigma + j\Omega = \frac{2}{T_d} \left[ \frac{2e^{-j\omega/2} j\sin(\omega/2)}{2e^{-j\omega/2} \cos(\omega/2)} \right] = \frac{2j}{T_d} \tan\left(\frac{\omega}{2}\right)
\]

- Which yields

\[
\Omega = \frac{2}{T_d} \tan\left(\frac{\omega}{2}\right) \quad \text{or} \quad \omega = 2 \arctan\left(\frac{\Omega T_d}{2}\right)
\]
Bilinear Transformation

\[ \omega = 2 \arctan \left( \frac{\Omega T_d}{2} \right) \]

\[ \Omega_p = \frac{2}{T_d} \tan \left( \frac{\omega_p}{2} \right) \]

\[ \Omega_s = \frac{2}{T_d} \tan \left( \frac{\omega_s}{2} \right) \]

\[ \Omega = \frac{2}{T_d} \tan \left( \frac{\omega}{2} \right) \]
Design method

1. Transformation with frequency warping
2. Transformation by lowpass prototype filter
3. Bilinear transformation

Digital filter transfer function and frequency response verification
Example

- **Bilinear transform applied to Butterworth**
  \[
  0.89125 \leq |H(e^{j\omega})| \leq 1 \quad 0 \leq |\omega| \leq 0.2\pi
  \]
  \[
  |H(e^{j\omega})| \leq 0.17783 \quad 0.3\pi \leq |\omega| \leq \pi
  \]

- **Apply bilinear transformation to specifications**
  \[
  0.89125 \leq |H(j\Omega)| \leq 1 \quad 0 \leq |\Omega| \leq \frac{2}{T_d} \tan\left(\frac{0.2\pi}{2}\right)
  \]
  \[
  |H(j\Omega)| \leq 0.17783 \quad \frac{2}{T_d} \tan\left(\frac{0.3\pi}{2}\right) \leq |\Omega| < \infty
  \]

- **We can assume** $T_d=1$ **and apply the specifications to**
  \[
  |H_c(j\Omega)|^2 = \frac{1}{1 + (\Omega / \Omega_c)^{2N}}
  \]

- **To get**
  \[
  1 + \left(\frac{2 \tan 0.1\pi}{\Omega_c}\right)^{2N} = \left(\frac{1}{0.89125}\right)^2 \quad \text{and} \quad 1 + \left(\frac{2 \tan 0.15\pi}{\Omega_c}\right)^{2N} = \left(\frac{1}{0.17783}\right)^2
  \]
Example Cont’d

- Solve $N$ and $\Omega_c$

$$N = \frac{\log\left(\left(\frac{1}{0.17783}\right)^2 - 1\right) / \left(\left(\frac{1}{0.89125}\right)^2 - 1\right)}{2 \log[\tan(0.15\pi)/\tan(0.1\pi)]} = 5.305 \approx 6 \quad \Omega_c = 0.766$$

- The resulting transfer function has the following poles

$$s_k = (-1)^{1/12}(j\Omega_c) = \Omega_c e^{(j\pi/12)(2k+1)} \quad \text{for } k = 0,1,...,11$$

- Resulting in

$$H_c(s) = \frac{0.20238}{(s^2 + 0.3996s + 0.5871)(s^2 + 1.0836s + 0.5871)(s^2 + 1.4802s + 0.5871)}$$

- Applying the bilinear transform yields

$$H(z) = \frac{0.0007378(1 + z^{-1})^6}{(1 - 1.2686z^{-1} + 0.7051z^{-2})(1 - 1.0106z^{-1} + 0.3583z^{-2})} \times \frac{1}{(1 - 0.9044z^{-1} + 0.2155z^{-2})}$$
Example Cont’d
Design example

- low pass filter
  - Specifications
    \[ 0.99 \leq |H(e^{j\omega})| \leq 1.01, \quad |\omega| \leq 0.4\pi, \]
    \[ |H(e^{j\omega})| \leq 0.001, \quad 0.6\pi \leq |\omega| \leq \pi. \]

- Butterworth filter
  - \( N=14 \)
Design example

- Chebyshev type I
  - $N=8$

- Chebyshev type II
  - $N=8$
Design example

- Elliptic filter
  - N=6

For a fixed specifications, the lowest order filter is obtained when the approximation error ripples equally between the extremes of the two bands.
Frequency transformation of lowpass IIR filters

- Design highpass, bandpass and bandstop filters
  - first design a low pass filter.
  - then using an algebraic transformation, derive the desired filter.
Table 11.2 Transformations from a discrete-time lowpass filter prototype with cutoff frequency $\theta_c$ to highpass, bandpass, and bandstop filters.

<table>
<thead>
<tr>
<th>Filter type</th>
<th>Transformation</th>
<th>Design parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lowpass</td>
<td>$z^{-1} \rightarrow \frac{z^{-1} - \alpha}{1 - \alpha z^{-1}}$</td>
<td>$\omega_c = \text{cutoff frequency}$ $\alpha = \frac{\sin((\theta_c - \omega_c)/2)}{\sin((\theta_c + \omega_c)/2)}$</td>
</tr>
<tr>
<td>Highpass</td>
<td>$z^{-1} \rightarrow -\frac{z^{-1} + \alpha}{1 + \alpha z^{-1}}$</td>
<td>$\omega_1 = \text{lower cutoff frequency}$ $\omega_2 = \text{upper cutoff frequency}$ $\alpha = \frac{\cos((\omega_2 + \omega_1)/2)}{\cos((\omega_2 - \omega_1)/2)}$ $K = \cot((\omega_2 - \omega_1)/2) \tan(\theta_c/2)$ $\alpha_1 = 2\alpha K/(K + 1)$ $\alpha_2 = (K - 1)/(K + 1)$</td>
</tr>
<tr>
<td>Bandpass</td>
<td>$z^{-1} \rightarrow -\frac{z^{-2} - \alpha_1 z^{-1} + \alpha_2}{\alpha_2 z^{-2} - \alpha_1 z^{-1} + 1}$</td>
<td>$\omega_1 = \text{lower cutoff frequency}$ $\omega_2 = \text{upper cutoff frequency}$ $\alpha = \frac{\cos((\omega_2 + \omega_1)/2)}{\cos((\omega_2 - \omega_1)/2)}$ $K = \tan((\omega_2 - \omega_1)/2) \tan(\theta_c/2)$ $\alpha_1 = 2\alpha/(K + 1)$ $\alpha_2 = (1 - K)/(1 + K)$</td>
</tr>
<tr>
<td>Bandstop</td>
<td>$z^{-1} \rightarrow \frac{z^{-2} - \alpha_1 z^{-1} + \alpha_2}{\alpha_2 z^{-2} - \alpha_1 z^{-1} + 1}$</td>
<td></td>
</tr>
</tbody>
</table>


FIR filter design
FIR filter

- Linear phase
- Window method
  - Simplest way of designing FIR filters
  - Start with ideal desired frequency response
  
  $$H_d(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h_d[n]e^{-j\omega n}$$
  $$h_d[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega})e^{j\omega n} d\omega$$

  - Ideal impulse responses are noncausal and of infinite length
  - The easiest way to obtain a causal FIR filter from ideal is to truncate the ideal impulse response.

  $$h[n] = \begin{cases} h_d[n] & 0 \leq n \leq M \\ 0 & \text{else} \end{cases}$$

  $$h[n] = h_d[n]w[n] \quad \text{where} \quad w[n] = \begin{cases} 1 & 0 \leq n \leq M \\ 0 & \text{else} \end{cases}$$
Windowing in frequency domain

- Windowed frequency response
  \[ H(e^{j\omega}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega})W(e^{j(\omega - \theta)})d\theta \]

- The windowed version is smeared version of desired response

- If \( w[n]=1 \) for all \( n \), then \( W(e^{j\omega}) \) is pulse train with \( 2\pi \) period
Properties of Windows

- It is desired that
  - $W(e^{j\omega})$ is concentrated in a narrow band of frequency to have less smearing.
  - $w[n]$ is as short as possible in duration to minimize computation in implementing of the filter.

- These are conflicting requirements!

- Example: Rectangular window

$$w[n] = \begin{cases} 
1 & 0 \leq n \leq M \\
0 & \text{else}
\end{cases}$$

$$W(e^{j\omega}) = \sum_{n=0}^{M} e^{-j\omega n} = 1 - e^{-j\omega(M+1)} \over 1 - e^{-j\omega} = e^{-j\omega M/2} \sin\left[\frac{\omega(M+1)/2}{2}\right] \over \sin\left[\frac{\omega}{2}\right]$$

- Peak sidelobe
- Mainlobe width

(M = 7)
Rectangular window

- Narrowest main lob
  - \( \frac{4\pi}{(M+1)} \)
  - Sharpest transitions at discontinuities in frequency
- Large side lobs
  - -13 dB
  - Large oscillation around discontinuities
- By tapering the window smoothly to zero at each end,
  - the height of the sidelobes can be diminished.
  - This is achieved at the expense of a wider mainlob and a wider transition at the discontinuity.
Bartlett (Triangular) Window

- Medium main lob
  - \(8\pi/M\)
- Side lobs
  - -25 dB
- Hamming window performs better

- Simple equation

\[
w[n] = \begin{cases} 
  2n / M & 0 \leq n \leq M / 2 \\
  2 - 2n / M & M / 2 \leq n \leq M \\
  0 & \text{else} 
\end{cases}
\]
Hanning Window

- Medium main lob
  - \(8\pi/M\)
- Side lobs
  - -31 dB
- Hamming window performs better
- Same complexity as Hamming

\[
w[n] = \begin{cases} 
\frac{1}{2} \left[ 1 - \cos \left( \frac{2\pi n}{M} \right) \right] & 0 \leq n \leq M \\
0 & \text{else}
\end{cases}
\]
Hamming Window

- Medium main lob
  - $8\pi/M$

- Good side lobes
  - -41 dB

- Simpler than Blackman

\[
w[n] = \begin{cases} 
0.54 - 0.46 \cos\left(\frac{2\pi n}{M}\right) & 0 \leq n \leq M \\
0 & \text{else}
\end{cases}
\]
Blackman Window

- Large main lob
  - $12\pi/M$

- Very good side lobs
  - -57 dB

- Complex equation

\[
w[n] = \begin{cases} 
0.42 - 0.5 \cos\left(\frac{2\pi n}{M}\right) + 0.08 \cos\left(\frac{4\pi n}{M}\right) & 0 \leq n \leq M \\
0 & \text{else}
\end{cases}
\]
Some properties

- Non-rectangular windows have wider main lobes and lower sidelobes.
- The width of transition band which is controlled by the width of the main lobe can be reduced by increasing the order $M$ of the filter.

<table>
<thead>
<tr>
<th>Type of Window</th>
<th>Peak Side-Lobe Amplitude (Relative)</th>
<th>Approximate Width of Main Lobe</th>
<th>Peak Approximation Error, $20 \log_{10} \delta$ (dB)</th>
<th>Equivalent Kaiser Window, $\beta$</th>
<th>Transition Width of Equivalent Kaiser Window</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rectangular</td>
<td>$-13$</td>
<td>$4\pi/(M + 1)$</td>
<td>$-21$</td>
<td>$0$</td>
<td>$1.81\pi/M$</td>
</tr>
<tr>
<td>Bartlett</td>
<td>$-25$</td>
<td>$8\pi/M$</td>
<td>$-25$</td>
<td>$1.33$</td>
<td>$2.37\pi/M$</td>
</tr>
<tr>
<td>Hanning</td>
<td>$-31$</td>
<td>$8\pi/M$</td>
<td>$-44$</td>
<td>$3.86$</td>
<td>$5.01\pi/M$</td>
</tr>
<tr>
<td>Hamming</td>
<td>$-41$</td>
<td>$8\pi/M$</td>
<td>$-53$</td>
<td>$4.86$</td>
<td>$6.27\pi/M$</td>
</tr>
<tr>
<td>Blackman</td>
<td>$-57$</td>
<td>$12\pi/M$</td>
<td>$-74$</td>
<td>$7.04$</td>
<td>$9.19\pi/M$</td>
</tr>
</tbody>
</table>
Figure 10.11 Magnitude responses in dB for 40th-order FIR lowpass filters designed using rectangular, Hann, Hamming, and Blackman windows with cutoff frequency $\omega_c = \pi/4$. 
Generalized Linear Phase

- It is desirable to obtain causal systems with linear phase.
- All windows are symmetric about point $M/2$.

$$w[n] = \begin{cases} w[M-n] & 0 \leq n \leq M \\ 0 & \text{else} \end{cases}$$

$$w_e[n] = w[n + M/2] \rightarrow W_e(e^{j\omega}) = W(e^{j\omega})e^{j\omega M/2} \Rightarrow W(e^{j\omega}) = W_e(e^{j\omega})e^{-j\omega M/2}$$

where $W_e(e^{j\omega})$ is a real and even function of $\omega \Rightarrow W(e^{j\omega})$ is linear phase.

If $h_d[M-n] = h_d[n] \Rightarrow h[n] = h_d[n]w[n] \Rightarrow \text{Linear phase}$

$$H(e^{j\omega}) = H_e(e^{j\omega})e^{-j\omega M/2}$$
Linear-Phase Lowpass filter

- Desired linear phase response

\[
H_{lp}(e^{j\omega}) = \begin{cases} 
  e^{-j\omega M/2} & |\omega| < \omega_c \\
  0 & \omega_c < |\omega| \leq \pi 
\end{cases}
\]

- Corresponding impulse response

\[
h_{lp}[n] = \frac{\sin[\omega_c(n - M/2)]}{\pi(n - M/2)}
\]

- Using a symmetric window, then a linear phase system will result.

\[
h[n] = \frac{\sin[\omega_c(n - M/2)]}{\pi(n - M/2)} w[n]
\]
Some notes

- Peak overshoot $\delta$.
- Peak undershoot $\delta$.
- Distance between the peak ripples is approximately the mainlobe width.

<table>
<thead>
<tr>
<th>Type of Window</th>
<th>Peak Side-Lobe Amplitude (Relative)</th>
<th>Approximate Width of Main Lobe</th>
<th>Peak Approximation Error, $20 \log_{10} \delta$ (dB)</th>
<th>Equivalent Kaiser Window, $\beta$</th>
<th>Transition Width of Equivalent Kaiser Window</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rectangular</td>
<td>-13</td>
<td>$4\pi/(M+1)$</td>
<td>-21</td>
<td>0</td>
<td>$1.81\pi/M$</td>
</tr>
<tr>
<td>Bartlett</td>
<td>-25</td>
<td>$8\pi/M$</td>
<td>-25</td>
<td>1.33</td>
<td>$2.37\pi/M$</td>
</tr>
<tr>
<td>Hanning</td>
<td>-31</td>
<td>$8\pi/M$</td>
<td>-44</td>
<td>3.86</td>
<td>$5.01\pi/M$</td>
</tr>
<tr>
<td>Hamming</td>
<td>-41</td>
<td>$8\pi/M$</td>
<td>-53</td>
<td>4.86</td>
<td>$6.27\pi/M$</td>
</tr>
<tr>
<td>Blackman</td>
<td>-57</td>
<td>$12\pi/M$</td>
<td>-74</td>
<td>7.04</td>
<td>$9.19\pi/M$</td>
</tr>
</tbody>
</table>
Design procedure

- Determine the cut-off frequency: 
  \[ w_c = \left( w_p + w_s \right) / 2 \]

- Determine 
  \[ A = -20 \log_{10} \delta, \quad \Delta w = w_s - w_p. \]

- From the table choose the window function that provides the smallest stopband attenuation greater than \( A \).

- For this window, determine the required value of \( M \) by selecting the corresponding value of \( \Delta w \).

- If \( M \) is odd, we may increase it by one to have flexible filter.

- Determine the impulse response of the ideal lowpass filter by
  \[ h_d[n] = \frac{\sin[\omega_c(n - M/2)]}{\pi(n - M/2)}. \]

- Compute
  \[ h[n] = h_d[n]w[n] \]
Example

- Specification: \( w_p = 0.25\pi, \ w_s = 0.35\pi, \ \delta = 0.0032 \)

  \[ \Rightarrow w_c = 0.3\pi, \quad \Delta \omega = 0.1\pi, \quad A = 50 \]

- Hamming window
- \( M = 80 \) approximately, \( M = 66 \) exactly
Typical application: speech noise reduction
Typical application: speech noise reduction

Filter type: lowpass FIR filter
Passband frequency range: 0–1,800 Hz
Passband ripple: 0.02 dB
Stopband frequency range: 2,000–4,000 Hz
Stopband attenuation: 50 dB

According to these specifications, we can determine the following parameters for filter design:

Window type = Hamming window
Number of filter taps = 133
Lowpass cutoff frequency = 1,900 Hz
Impulse response of standard FIR filters

- **Low pass**

  Low pass:  \( h[n] = \begin{cases} \frac{\omega_c}{\pi} & n = 0 \\ \sin(\omega_c n) & n \neq 0 \end{cases} \)

- **High pass**

  High pass:  \( h[n] = \begin{cases} 1 - \frac{\omega_c}{\pi} & n = 0 \\ -\frac{\sin(\omega_c n)}{\pi n} & n \neq 0 \end{cases} \)

- **Bandpass**

  Band pass:  \( h[n] = \begin{cases} \frac{\omega_H - \omega_L}{\pi} & n = 0 \\ \frac{\sin(\omega_H n)}{\pi n} - \frac{\sin(\omega_L n)}{\pi n} & n \neq 0 \end{cases} \)

- **Bandstop**

  Band stop:  \( h[n] = \begin{cases} 1 - \frac{\omega_H - \omega_L}{\pi} & n = 0 \\ -\frac{\sin(\omega_H n)}{\pi n} + \frac{\sin(\omega_L n)}{\pi n} & n \neq 0 \end{cases} \)
Kaiser Window Filter

- Parameterized equation forming a set of windows
  - Parameter to change main-lob width and side-lob area trade-off

\[
w[n] = \begin{cases} 
I_0 \left( \beta \sqrt{1 - \left( \frac{n - M/2}{M/2} \right)^2} \right) & 0 \leq n \leq M \\
I_0(\beta) & \text{else} \\
0 & \text{else}
\end{cases}
\]

- \(I_0(\cdot)\) represents zeroth-order modified Bessel function of \(1^{st}\) kind
Determining Kaiser Window Parameters

- Given filter specifications Kaiser developed empirical equations
  - Given the peak approximation error $\delta$ or in dB as $A = -20\log_{10} \delta$
  - and transition band width $\Delta \omega = \omega_s - \omega_p$
- The shape parameter $\beta$ should be
  
  \[
  \beta = \begin{cases} 
  0.1102(A - 8.7) & A > 50 \\
  0.5842(A - 21)^{0.4} + 0.07886(A - 21) & 21 \leq A \leq 50 \\
  0 & A < 21 
  \end{cases}
  \]

- The filter order $M$ is determined approximately by
  
  \[
  M = \frac{A - 8}{2.285\Delta \omega}
  \]
Example: Kaiser Window Design of a Lowpass Filter

- Specifications
  \[ \omega_p = 0.4\pi, \omega_p = 0.6\pi, \delta_1 = 0.01, \delta_2 = 0.001 \]

- Window design methods assume
  \[ \delta_1 = \delta_2 = 0.001 \]

- Determine cut-off frequency
  - Due to the symmetry we can choose it to be \( \omega_c = 0.5\pi \)

- Compute
  \[ \Delta \omega = \omega_s - \omega_p = 0.2\pi \]
  \[ A = -20 \log_{10} \delta = 60 \]

- And Kaiser window parameters
  \[ \beta = 5.653 \]
  \[ M = 37 \]

- Then the impulse response is given as
  \[ h[n] = \begin{cases} 
  \frac{\sin[0.5\pi(n - 18.5)]}{\pi(n - 18.5)} & \text{for } 0 \leq n \leq M \\
  \frac{I_0(5.653)}{I_0(5.653)} & \text{else}
  \end{cases} \]
Example Cont’d

Approximation Error

Graphs showing sample number versus amplitude and radian frequency versus amplitude for approximation error.
Example

- Specification  \( w_p = 0.25\pi, \ w_s = 0.35\pi, \ \delta = 0.0032 \)

\[ \Rightarrow w_c = 0.3\pi, \ \Delta\omega = 0.1\pi, \ A = 50 \]

\[ \Rightarrow \beta = 4.5, \ M = 59 \approx 60 \]
General Frequency Selective Filters

- A general multiband impulse response can be written as

\[ h_{mb}[n] = \sum_{k=1}^{N_{mb}} (G_k - G_{k+1}) \frac{\sin \omega_k(n - M/2)}{\pi(n - M/2)} \]

- Window methods can be applied to multiband filters
- Example multiband frequency response

- Special cases of
  - Bandpass
  - Highpass
  - Bandstop
Filter design tool
References

- Discrete-Time Signal Processing, 2e by Oppenheim, Shafer
- Lecture notes by Güner Arslan Dept. of Electrical and Computer Engineering, The University of Texas at Austin