Chapter 3: Mathematical modeling of dynamic systems

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Mathematical modeling of dynamic systems:

- **Simplicity versus accuracy**: it is possible to improve the accuracy of mathematical model by increasing its complexity.

- **Linear systems**: a system is called linear if principle of superposition applies.

- **Linear time invariant systems and linear time variant systems**: a differential equation is linear if the coefficients are constants or of the independent variable.

Examples of Nonlinear Systems:

\[
\frac{d^2 x}{dt^2} + \left( \frac{dx}{dt} \right)^2 + x = A \sin(\omega t)
\]

\[
\frac{d^2 x}{dt^2} + (x^2 - 1) \frac{dx}{dt} + x = 0
\]

\[
\frac{d^2 x}{dt^2} + \frac{dx}{dt} + x + x^3 = 0
\]

Linearization of nonlinear systems:

- In control engineering a normal operation of the system may be around an equilibrium point, and the signals may be considered small signals around equilibrium.

- If the system operates around an equilibrium point and if signals involved are small signals, then it is possible to approximate the nonlinear system by a linear system.
Transfer function and impulse-response function:

- In control theory, functions called transfer functions are commonly used to characterize the input-output relationships of components or systems that can be described by linear, time invariant, differential equations.

- The transfer functions of a linear, time invariant, differential equation is defined as the ratio of the Laplace transform of the output to the Laplace transform of the input under the assumption that all initial conditions are zero.

Example: Consider the satellite attitude control system shown in the figure, the transfer function is determined as follows

\[ J \frac{d^2 \theta}{dt} = T \]
\[ J s^2 \Theta(s) = T(s) \]
\[ \frac{\Theta(s)}{T(s)} = \frac{1}{s^2} \]

Convolution Integral:

- For a linear, time invariant system transfer function is

\[ G(s) = \frac{Y(s)}{X(s)} \]

- Where \( X(s) \) is the Laplace transform of the input and \( Y(s) \) is the Laplace transform of the output, where we assume that all initial conditions involved are zero, hence:

\[ Y(s) = G(s) X(s) \]

- So the inverse Laplace function is given by the following convolution integral:

\[ y(t) = \int_0^t x(\tau) g(t-\tau) d\tau \]

\[ = \int_0^t g(t-\tau) x(\tau) d\tau \]

Impulse-response function:

- Consider the output of a system to a unit-impulse input when the initial conditions are zero. Since the Laplace transform function of unit impulse function is unity, the Laplace transform of the output of the system is:

\[ Y(s) = G(s) \]

- Thus the inverse Laplace transform of the output is given by:

\[ y(t) = L^{-1}[G(s)] \]
This diagram shows a pressure controller. $y=0$ is the equilibrium position of the device.

At $t=0$ there is a sudden increase in pressure. The motion is governed by:

$$m\ddot{y} + c\dot{y} + ky = Pu(t)$$

Find $y(t)$.
(P is the force due to the applied pressure).

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![Diagram of a pressure controller with a membrane and spring.](image)

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**Modelling Example**

- First transform the ODE:

$$m[s^2Y - sy(0) - y'(0)] + c[sY - y(0)] + kY = P \frac{e^{-0s}}{s} = \frac{P}{s}$$

- The system is initially at rest so this simplifies to:

$$s^2mY + scY + kY = \frac{P}{s}$$

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**Modelling Example**

- Now solve for $Y$:

$$Y = P \frac{1}{s(s^2m + sc + k)}$$

- Now the hard part ... we must invert this for $y(t)$. Since $Y$ is in the form of one polynomial divided by another we will expand it using partial fractions.

$$Y = P \left( \frac{c_1}{s} + \frac{c_2s + c_3}{s^2m + sc + k} \right)$$

---

**Modelling Example**

- Determining unknown coefficients we have:

$$Y = P \left( \frac{1}{ks} - \frac{m}{k} \frac{s + c}{s^2m + sc + k} \right)$$

... which can at least be inverted term by term.
Modelling Example

• From a table of Laplace transforms we have:

\[
L^{-1}\left\{\frac{s}{(s-a)(s-b)}\right\} = \frac{1}{(a-b)}(ae^{at} - be^{bt})
\]

\[
L^{-1}\left\{\frac{1}{(s-a)(s-b)}\right\} = \frac{1}{(a-b)}(e^{at} - e^{bt})
\]

Modelling Example

• Putting every together we have:

\[
y(t) = \frac{P}{k}\left[1 - e^{-\alpha t}\left(\frac{1}{k} \cos(\omega t) + \frac{c}{\omega m k} \sin(\omega t)\right)\right]
\]

From first partial fraction

From real part of the second partial fraction.

Block diagram

• A system may consist of a number of components. To show the functions performed by each component, we commonly use a diagram called the block diagram.

Block diagram of a close loop system.

- Summing point
- Branch point
- Transfer function \( G(s) \)
- Summing point
- \( R(s) \)
- \( E(s) \)
- \( G(s) \)
- \( C(s) \)
Open loop transfer function and feedforward transfer function:

Feedback signal = \( B(s) = H(s)C(S) \)

Open loop transfer function = \( \frac{B(s)}{E(s)} = G(s)H(s) \)

Feed Forward transfer function = \( \frac{C(s)}{E(s)} = G(s) \)

Closed-loop transfer function:

\[
C(s) = G(s) E(S) \\
E(s) = R(s) - B(s) = R(s) - H(s)C(s)
\]

Eliminating \( E(s) \) from the above equation yields:

\[
C(s) = G(s)\left[ R(s) - H(s)C(s) \right]
\]

Thus close loop transfer function is obtained as:

\[
\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}
\]

Example: consider the RC circuit shown in the figure, using the system dynamic equations, the overall block diagram of the system can be obtained as it is drown:

\[
i = \frac{e_i - e_o}{R} \quad I(s) = \frac{E_i(s) - E_o(s)}{R}
\]

\[
e_o = \frac{1}{C} \int i \, dt \quad E_o(s) = \frac{I(s)}{Cs}
\]
Rules of Block Diagram Algebra

Example: the block diagram shown in figure (a) can be simplified step by step as it is shown in figures (b) through (e).

Example: simplify the block diagram shown.
Classical or Frequency-Domain Technique:

- **Advantages**
  - Converts differential equation into algebraic equation via transfer functions.
  - Rapidly provides stability & transient response info.

- **Disadvantages**
  - Applicable only to Linear, Time-Invariant (LTI) systems or their close approximations.

The limitation became a problem circa 1960 when space applications became important.

Modeling in the Time Domain - State-Space:

- **State-Space or Modern or Time-Domain technique**

- **Advantages**
  - Provides a unified method for modeling, analyzing, and designing a wide range of systems using matrix algebra.
  - Nonlinear, Time-Varying, Multivariable systems

- **Disadvantages**
  - Not as intuitive as classical method.
  - Calculations required before physical interpretation is apparent

State-Space Representation

An LTI system is represented in state-space format by the vector-matrix differential equation (DE) as:

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) & \text{Dynamic equation(s)} \\
y(t) &= Cx(t) + Du(t) & \text{Measurement equations}
\end{align*}
\]

with \( t \geq t_0 \) and initial conditions \( x(t_0) \).

Definitions

- **System variables**: Any variable that responds to an input or initial conditions.

- **State variables**: The smallest set of linearly independent system variables such that the initial condition set and applied inputs completely determine the future behavior of the set.

Linear Independence: A set of variables is linearly independent if none of the variables can be written as a linear combination of the others.
Definitions

- **State vector**: An \((n \times 1)\) column vector whose elements are the state variables.

- **State space**: The \(n\)-dimensional space whose axes are the state variables.

The minimum number of state variables is equal to:

- the order of the DE’s describing the system.
- the order of the denominator polynomial of its transfer function model.
- the number of independent energy storage elements in the system.

Remember the state variables must be linearly independent! If not, you may not be able to solve for all the other system variables, or even write the state equations.

In General:

\[
\dot{x}(t) = f(x, u, t) \\
y(t) = g(x, u, t)
\]

\[
x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}, \quad f(x, u, t) = \begin{bmatrix} f_1(x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_r, t) \\ f_2(x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_r, t) \\ \vdots \\ f_n(x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_r, t) \end{bmatrix} \\
y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{bmatrix}, \quad g(x, u, t) = \begin{bmatrix} g_1(x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_r, t) \\ g_2(x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_r, t) \\ \vdots \\ g_m(x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_r, t) \end{bmatrix}, \quad u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_r(t) \end{bmatrix}
\]

Linearization

\[
y = f(x) \\
= f(\bar{x}) + \frac{df}{dx} (x - \bar{x}) + \frac{1}{2!} \frac{d^2f}{dx^2} (x - \bar{x})^2 + \cdots
\]

\[
y - \bar{y} = K(x - \bar{x})
\]

\[
\bar{y} = f(\bar{x}) \quad K = \left. \frac{df}{dx} \right|_{x=\bar{x}}
\]
Example:

\[ m\ddot{y} + b\dot{y} + ky = u \]

\[ x_1(t) = y(t) \]

\[ x_2(t) = \dot{y}(t) \]

\[
\begin{bmatrix}
  \dot{x}_1 \\
  \dot{x}_2
\end{bmatrix} = \begin{bmatrix}
  0 & 1 \\
  -\frac{k}{m} & -\frac{b}{m}
\end{bmatrix} \begin{bmatrix}
  x_1 \\
  x_2
\end{bmatrix} + \begin{bmatrix}
  0 \\
  1/m
\end{bmatrix} u
\]

Converting a Transfer Function to State Space

- State variables are not unique. A system can be accurately modeled by several different sets of state variables.
- Sometimes the state variables are selected because they are physically meaningful.
- Sometimes because they yield mathematically tractable state equations.
- Sometimes by convention.

Phase-variable Format

1. Consider the DE

\[
\frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_0 y = b_0 u
\]

where \( y \) is the measure variable and \( u \) is the input.

2. The minimum number of state variables is \( n \) since the DE is of nth order.

3. Choose the output and its derivatives as state variables.

\[
x_1 = y \\
x_2 = \dot{y} \Rightarrow \dot{x}_1 = x_2 \\
\vdots \\
x_n = \frac{d^{n-1} y}{dt^{n-1}} \Rightarrow \dot{x}_{n-1} = x_n \\
x_n = \frac{d^n y}{dt^n} \Rightarrow \dot{x}_n = x_n + b_0 u
\]

First row of state equations

Last row of state equations

Converting from State Space to a Transfer Function

\[
\dot{x}(t) = Ax(t) + Bu(t) \\
y(t) = Cx(t) + Du(t)
\]

with \( t \geq t_0 \) and zero initial conditions.

Taking the Laplace transform,

\[
x(s) = X(s) = [sI - A]^{-1} Bu(s)
\]

\[
y(s) = Y(s) = Cx(s) + Du(s) = C[sI - A]^{-1} Bu(s) + Du(s)
\]

\[
= [C[sI - A]^{-1} B + D]u(s)
\]

In the case of SISO (Single - Input, Single - Output) systems:

\[
\frac{Y(s)}{U(s)} = C[sI - A]^{-1} B + D = \frac{C \text{ adj}[sI - A] B + \det[sI - A] D}{\det[sI - A]}
\]
Phase-variable Format

4. Arrange in vector-matrix format

\[
\begin{bmatrix}
    \frac{d}{dt} x_1 \\
    \frac{d}{dt} x_2 \\
    \vdots \\
    \frac{d}{dt} x_n \\
\end{bmatrix}
= \begin{bmatrix}
    0 & 1 & 0 & \ldots & 0 \\
    0 & 0 & 1 & \ldots & 0 \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & 0 & \ldots & 1 \\
\end{bmatrix} \begin{bmatrix}
    x_1 \\
    x_2 \\
    \vdots \\
    x_n \\
\end{bmatrix} + \begin{bmatrix}
    b_0 \\
    b_1 \\
    \vdots \\
    b_n \\
\end{bmatrix}
\]

\[
y = \begin{bmatrix}
    1 & 0 & 0 & \cdots & 0 \\
\end{bmatrix} \begin{bmatrix}
    x_1 \\
    x_2 \\
    \vdots \\
    x_n \\
\end{bmatrix} = \begin{bmatrix}
    y_1 \\
    y_2 \\
    \vdots \\
    y_n \\
\end{bmatrix}
\]

Note the transfer function format

\[
\frac{Y(s)}{U(s)} = \frac{b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1s + a_0}
\]

Example:

```
\[R(s) \rightarrow \frac{b_2s^2 + b_1s + b_0}{a_3s^3 + a_2s^2 + a_1s + a_0} \rightarrow C(s)\]
```

Transfer Function with Numerator Polynomial (continued)

1. From the first block:

\[
\frac{X_1(s)}{R(s)} = \frac{1/a_3}{s^3 + a_2s^2 + a_1s + a_0} = \frac{1/a_3}{s + a_2/a_3 + a_1/a_3 + a_0/a_3}
\]

2. Therefore,

\[
\frac{d}{dt} \begin{bmatrix}
    x_1 \\
    x_2 \\
    x_3 \\
\end{bmatrix} = \begin{bmatrix}
    0 & 1 & 0 \\
    0 & 0 & 1 \\
    -a_0/a_3 & -a_1/a_3 & -a_2/a_3
\end{bmatrix} \begin{bmatrix}
    x_1 \\
    x_2 \\
    x_3 \\
\end{bmatrix} + \begin{bmatrix}
    0 \\
    0 \\
    0
\end{bmatrix} u
\]
Transfer Function with Numerator Polynomial (continued)

3. The measurement (observation) equation is obtained from the second transfer function.

\[ C(s) = Y(s) = \left(b_2 s^2 + b_1 s + b_0\right)X_1(s) = b_2 s^2 X_1 + b_1 s X_1 + b_0 X_1 \]

But, \( sX_1 = X_2 \) and \( sX_2 = X_3 \)

So, \( Y(s) = b_2 X_3 + b_1 X_2 + b_0 X_1 \)

\[ y = \begin{bmatrix} [b_0 & b_1 & b_2] \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \]

If forcing function involve derivative terms:

\[
\begin{align*}
(n) & +(n-1) \quad + \cdots + a_{n-1} \dot{y} + a_n y = b_0 \dot{u} + b_1 u + \cdots + b_{n-1} \dot{u} + b_n u \\
\end{align*}
\]

\[
\begin{align*}
x_1 &= y - \beta_0 \dot{u} \\
x_2 &= \dot{y} - \beta_0 \ddot{u} - \beta_1 \dot{u} = \dot{x}_1 - \beta_1 u \\
x_3 &= \ddot{y} - \beta_0 \ddot{u} - \beta_1 \dot{u} - \beta_2 \ddot{u} = \ddot{x}_2 - \beta_2 \ddot{u} \\
&\quad \cdots \\
x_n &= y - \beta_0 \ddot{u} - \beta_1 \dot{u} - \cdots - \beta_{n-2} \ddot{u} - \beta_{n-1} u = \ddot{x}_{n-1} - \beta_{n-1} u \\
\end{align*}
\]

Example:

\[
\begin{align*}
\frac{1}{s^3 + 9s^2 + 26s + 24} & \quad \frac{s^2 + 7s + 2}{s^3 + 9s^2 + 26s + 24} \\
\end{align*}
\]

Internal variables:
\( \lambda_2(t), \lambda_3(t) \)

\[
\begin{align*}
\beta_0 &= b_0 \\
\beta_1 &= b_1 - a_1 \beta_0 \\
\beta_2 &= b_2 - a_1 \beta_1 - a_2 \beta_0 \\
\beta_3 &= b_3 - a_1 \beta_2 - a_2 \beta_1 - a_3 \beta_0 \\
&\quad \cdots \\
\beta_n &= b_n - a_1 \beta_{n-1} - \cdots - a_{n-1} \beta_1 - a_n \beta_0 \\
\end{align*}
\]

\[
\begin{align*}
\dot{x}_1 &= x_2 + \beta_1 u \\
\dot{x}_2 &= x_3 + \beta_2 u \\
&\quad \cdots \\
\dot{x}_{n-1} &= x_n + \beta_{n-1} u \\
\dot{x}_n &= -a_n x_1 - a_{n-1} x_2 - \cdots - a_1 x_n + \beta_n u \\
\end{align*}
\]
Mechanical systems

The fundamental law governing mechanical systems is Newton's second law

Example

Transfer Function

\[ m \frac{d^2y}{dt^2} = -b\left(\frac{dy}{dt} - \frac{du}{dt}\right) - k(y - u) \]

\[ m \frac{d^2y}{dt^2} + b \frac{dy}{dt} + ky = b \frac{du}{dt} + ku \]

\[(ms^2 + bs + k)Y(s) = (bs + k)U(s)\]

\[ Y(s) = \frac{bs + k}{ms^2 + bs + k} \]

State-space model

\[ \ddot{y} + \frac{b}{m} \dot{y} + \frac{k}{m} y = \frac{b}{m} \dot{u} + \frac{k}{m} u \]

Comparing with standard form:

\[ \ddot{y} + a_1 \dot{y} + a_2 y = b_0 \dot{u} + b_1 \dot{u} + b_2 u \]

gives

\[ a_1 = \frac{b}{m}, \quad a_2 = \frac{k}{m}, \quad b_0 = 0, \quad b_1 = \frac{b}{m}, \quad b_2 = \frac{k}{m} \]

\[ \beta_0 = b_0 = 0 \]

\[ \beta_1 = b_1 - a_1 \beta_0 = \frac{b}{m} \]

\[ \beta_2 = b_2 - a_1 \beta_1 - a_2 \beta_0 = \frac{k}{m} - \left(\frac{b}{m}\right)^2 \]
\[ x_1 = y - \beta \mu = y \]
\[ x_2 = \dot{x}_1 - \beta_1 u = \dot{x}_1 - \frac{b}{m} u \]
\[ \dot{x}_1 = x_2 + \beta_1 u = x_2 + \frac{d}{m} u \]
\[ \dot{x}_2 = -a_2 x_1 - a_1 x_2 + \beta_2 u = -\frac{k}{m} x_1 - \frac{b}{m} x_2 + \left[ \frac{k}{m} - \frac{(b/m)^2}{m} \right] u \]
\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} = 
\begin{bmatrix}
0 & 1 \\
-k/m & -b/m
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} + 
\begin{bmatrix}
b/m \\
k/m - (b/m)^2
\end{bmatrix} u
\]
\[ y = [1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \]

**Example**

![Example Diagram]

**Electrical Systems**

Basic laws governing electrical circuits are Kirchhoff’s current law and voltage law.

**Kirchhoff’s current law:** The algebraic sum of all currents entering and leaving a node is zero

**Kirchhoff’s voltage law:** The algebraic sum of the voltages around any loop in an electrical circuit is zero

\[ I\ddot{\theta} = Vl \sin \theta - Hl \cos \theta \]
\[ m \frac{d^2}{dt^2} (x + l \sin \theta) = H \]
\[ m \frac{d^2}{dt^2} (l \cos \theta) = V - mg \]
\[ M \frac{d^2}{dt^2} x = u - H \]

\[ (M + m)\ddot{x} + ml\ddot{\theta} = u \]
\[ (l + ml^2)\ddot{\theta} + ml\ddot{x} = mgl\theta \]
Example: RLC circuit

\[ L \frac{di}{dt} + Ri + \frac{1}{C} \int i \, dt = e_i \]
\[ \frac{1}{C} \int i \, dt = e_o \]

\[ \frac{E_o(s)}{E_i(s)} = \frac{1}{LCs^2 + RCs + 1} \]

Complex impedances:

\[ Z(s) = E(s)/I(s) \]

- In driving transfer functions for electrical circuits, we frequently find it convenient to write Laplace transformed equations directly, without writing differential equations.

- The complex Impedance \( Z(s) \) of a two terminal circuit is the ratio of \( E(s) \), the Laplace transform of the voltage across the elements to \( I(s) \), the Laplace transform of the current through the element.

- If the two terminal element is a resistance \( R \), capacitance \( C \), or inductance \( L \), then the complex impedance is given by \( R \), \( 1/Cs \), or \( Ls \) respectively.

- If the complex impedances are connected in series, the total impedance is the sum of individual complex impedances.

Example

\[ \frac{E_o(s)}{E_i(s)} = \frac{Z_2(s)}{Z_1(s) + Z_2(s)} \]

\[ Z_1 = Ls + R, \quad Z_2 = \frac{1}{Cs} \]

Example

\[ \frac{1}{C_1s} [I_1(s) - I_2(s)] + R_1I_1(s) = E_i(s) \]
\[ \frac{1}{C_1s} [I_2(s) - I_1(s)] + R_2I_2(s) + \frac{1}{C_2s} I_2(s) = 0 \]
\[ \frac{1}{C_2s} I_2(s) = E_o(s) \]
\[ \frac{E_o(s)}{E_i(s)} = \frac{1}{(R_1C_1s + 1)(R_2C_2s + 1) + R_1C_2s} \]
Liquid level systems

For laminar flow \[ Q = KH \]
where \( Q \) = steady-state liquid flow rate, m³/sec
\( K \) = coefficient, m²/sec
\( H \) = steady-state head, m

\[ R_l = \frac{dH}{dQ} = \frac{H}{Q} \]

For turbulent flow \[ Q = K \sqrt{H} \]
where \( Q \) = steady-state liquid flow rate, m³/sec
\( K \) = coefficient, m²/sec
\( H \) = steady-state head, m

Slope of curve at point \( P = \frac{h}{q} = \frac{2\sqrt{H}}{Q} = R_t \)

Example

\[ C \, dh = (q_i - q_o) \, dt \]
\[ q_o = \frac{h}{R} \]
\[ RC \, \frac{dh}{dt} + h = R q_i \]
\[ (RC + 1)H(s) = RQ_i(s) \]
\[ \frac{Q_o(s)}{Q_i(s)} = \frac{1}{RC + 1} \]

Example

\[ \frac{h_1 - h_2}{R_1} = q_1 \]
\[ \frac{h_2}{R_2} = q_2 \]
\[ C_1 \frac{dh_1}{dt} = q - q_1 \]
\[ C_2 \frac{dh_2}{dt} = q_1 - q_2 \]
\[ \frac{Q_2(s)}{Q(s)} = \frac{1}{R_1 C_1 R_2 C_2 s^2 + (R_1 C_1 + R_2 C_2 + R_2 C_1)s + 1} \]

Example

\[ \frac{h_1}{R_1} = q_1 \]
\[ \frac{h_2}{R_2} = q_2 \]
\[ C_1 \frac{dh_1}{dt} = q - q_1 \]
\[ C_2 \frac{dh_2}{dt} = q_1 - q_2 \]
\[ \frac{Q_2(s)}{Q(s)} = \frac{1}{R_1 C_1 R_2 C_2 s^2 + (R_1 C_1 + R_2 C_2 + R_2 C_1)s + 1} \]
Thermal systems

• Thermal systems are those that involve the transfer of heat from one substance to another.

• To simplify the analysis we assume that a thermal system can be represented by a lumped parameter model, that substances that are characterized by resistance to heat flow have negligible heat capacitance and that substances that are characterized by heat capacitance have negligible resistance to heat flow.

• Here we only consider conduction and convection.

For conduction or convection heat transfer,

\[ q = K \Delta \theta \]

where \( q \) = heat flow rate, kcal/sec
\( \Delta \theta \) = temperature difference, °C
\( K \) = coefficient, kcal/sec °C

The coefficient \( K \) is given by

\[ K = \frac{kA}{\Delta X} , \quad \text{for conduction} \]
\[ = HA, \quad \text{for convection} \]

where \( k \) = thermal conductivity, kcal/m sec °C
\( A \) = area normal to heat flow, m²
\( \Delta X \) = thickness of conductor, m
\( H \) = convection coefficient, kcal/m² sec °C
Thermal resistance and thermal capacitance

\[ R = \frac{\text{change in temperature difference}, ^\circ\text{C}}{\text{change in heat flow rate}, \text{kcal/sec}} \]
\[ C = \frac{\text{change in heat stored}, \text{kcal}}{\text{change in temperature}, ^\circ\text{C}} \]

\[ R = \frac{d(\Delta \theta)}{dq} = \frac{1}{K} \quad \text{for conduction or convection} \]
\[ C = mc \]

where \( m \) = mass of substance considered, kg
\( c \) = specific heat of substance, kcal/kg \(^\circ\text{C}\)

Example

\( \dot{\theta}_i \) = steady-state temperature of inflowing liquid, \(^\circ\text{C}\)
\( \dot{\theta}_o \) = steady-state temperature of outflowing liquid, \(^\circ\text{C}\)
\( G \) = steady-state liquid flow rate, kg/sec
\( M \) = mass of liquid in tank, kg
\( c \) = specific heat of liquid, kcal/kg \(^\circ\text{C}\)
\( R \) = thermal resistance, \(^\circ\text{C}\) sec/kcal
\( C \) = thermal capacitance, kcal/\(^\circ\text{C}\)
\( \dot{H} \) = steady-state heat input rate, kcal/sec

Linearized servo hydraulic system

Around normal operating point (\( x = 0, \Delta \rho = 0, \dot{q} = 0 \)):

\[ q = K_1 x - K_2 \Delta p \]